Unit - IV

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Basically *Vector* is a quantity having both magnitude and direction. Vector quantities like force, velocity, acceleration etc. have lot of reference in physical and engineering problems. We are familiar with vector algebra which gives an exposure to all the basic concepts related to vectors.

Differentiation and Integration are well acquainted topics in calculus. In the background of all these we discuss this chapter *Vector Calculus* comprising *Vector Differentiation*. Many concepts are highly significant in various branches of engineering.

[illegible] derivative of a [illegible]

Let the position vector of a point $P(x, y, z)$ in space be

$$\vec{r} = x\,i + y\,j + z\,k$$

If x, y, z are all functions of a single parameter t, then $\vec{r}$ is said to be a vector function of t which is also referred to as a *vector point function* usually denoted as $\vec{r} = \vec{r}(t)$. As the parameter t varies, the point P traces a curve in space. Therefore

$$\vec{r} = x(t)\, i + y(t)\, j + z(t)\, k$$

is called as the vector equation of the curve.

$$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \frac{dx}{dt}\, i + \frac{dy}{dt}\, j + \frac{dz}{dt}\, k$$

is a **vector along the tangent to the curve at** P.

If t is the time variable,

$\vec{v} = \dfrac{d\vec{r}}{dt}$ gives the **velocity of the particle at time** t.

Further $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$ represents the rate of change of velocity $\vec{v}$ and is called the **acceleration of the particle at time** t.

Properties

1. $\frac{d}{dt}\left\{c_1\vec{r}_1(t) \pm c_2\vec{r}_2(t)\right\} = c_1\vec{r}_1{}'(t) \pm c_2\vec{r}_2{}'(t)$ where c_1, c_2 are constants.
2. $\frac{d}{dt}(\vec{F}\cdot\vec{G}) = \vec{F}\cdot\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\cdot\vec{G}$
3. $\frac{d}{dt}(\vec{F}\times\vec{G}) = \vec{F}\times\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\times\vec{G}$

where $\vec{F} = \vec{F}(t)$ and $\vec{G} = \vec{G}(t)$.

If to every point (x, y, z) of a region R in space there corresponds a scalar $\phi(x, y, z)$ then ϕ is called a *scalar point function* and we say that a *scalar field* ϕ is defined in R.

Examples : 1. $\phi = x^2 + y^2 + z^2$ 2. $\phi = x y^2 z^3$

If to every point (x, y, z) of a region R in space there corresponds a vector $\vec{A}(x, y, z)$ then $\vec{A}$ is called a *vector point function* and we say that a *vector field* $\vec{A}$ is defined in R.

Examples : 1. $\vec{A} = x^2 i + y^2 j + z^2 k$ 2. $\vec{A} = xyz\, i + yz\, j + z\, k$

Operators (i) The *vector differential operator* ∇, *read as "Nabla" or "Del"* is defined by

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \Sigma \frac{\partial}{\partial x} i$$

(ii) The *Laplacian operator* ∇^2 is defined by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Sigma \frac{\partial^2}{\partial x^2}$$

We now proceed to define four important quantities associated with the operators ∇ and ∇^2.

If $\phi(x, y, z)$ is a continuously differentiable scalar function then the **gradient of ϕ** (grad ϕ *in precise*) is defined to be $\nabla\phi$.

ie., $$\text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

Obviously $\nabla\phi$ is a vector quantity.

If $\vec{A}(x, y, z)$ is a continuously differentiable vector function then **divergence of** $\vec{A}$ (div $\vec{A}$ *in precise*) is defined to be $\nabla \cdot \vec{A}$

If $\vec{A} = a_1 i + a_2 j + a_3 k$, where a_1, a_2, a_3 are all functions of x, y, z then we have

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot (a_1 i + a_2 j + a_3 k)$$

ie., $$\text{div } \vec{A} = \nabla \cdot \vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

Clearly div $\vec{A}$ is a scalar quantity.

If $\vec{A}(x, y, z)$ is a continuously differentiable vector function then **curl of** $\vec{A}$ (curl $\vec{A}$ *in precise*) is defined to be $\nabla \times \vec{A}$

If $\vec{A} = a_1 i + a_2 j + a_3 k$ where a_1, a_2, a_3 are all functions of x, y, z then we have

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) - j \left(\frac{\partial a_3}{\partial x} - \frac{\partial a_1}{\partial z} \right) + k \left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right)$$

Clearly curl $\vec{A}$ is a vector quantity.

Laplacian : If $\phi(x, y, z)$ is a continuously differentiable scalar function and $\vec{A}(x, y, z)$ is a continuously differentiable vector function we can define the Laplacian for ϕ as well as for $\vec{A}$ as follows.

Laplacian of $\phi = \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$

Laplacian of $\vec{A} = \nabla^2 \vec{A} = \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2}$

Note : *If ϕ is a scalar function, the equation $\nabla^2 \phi = 0$ is called Laplace's equation and a function which satisfies Laplace's equation is called a harmonic function.*

Also $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ *is called Laplace's equation in two dimensions.*

Obviously Laplacian of a scalar function is a scalar quantity and Laplacian of a vector function is a vector quantity.

Remark : **1.** *If $\phi(x, y, z)$ is a scalar function then we have*

$$\text{grad}\phi = \nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$\therefore$ $\quad$ $\text{div}(\text{grad}\,\phi) = \nabla \cdot \nabla\phi$

ie.,
$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi$$

Thus $\quad$ **div (grad ϕ) = $\nabla^2 \phi$ or $\nabla \cdot \nabla \phi = \nabla^2 \phi$**

2. *The computation of gradient, divergence, curl and laplacian is a combination of the concepts of vector algebra and partial differentiation.*

Here are a few illustrations.

1. Given $\phi = x^2 y + y^2 z + z^2 x$ let us find $\nabla \phi$ and $\nabla^2 \phi$

$$\nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

ie., $\quad$ $\nabla\phi = (2xy + z^2)\, i + (x^2 + 2yz)\, j + (y^2 + 2xz)\, k$

Next $\quad$ $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ or $\nabla \cdot (\nabla\phi)$.

ie., $\quad$ $\nabla^2\phi = 2y + 2z + 2x = 2(x + y + z)$

2. Given $\vec{A} = x^2yz\,i + y^2zx\,j + z^2xy\,k$ let us find div $\vec{A}$, curl $\vec{A}$ and $\nabla^2\vec{A}$.

$$\text{div } \vec{A} = \nabla \cdot \vec{A}$$
$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(x^2yz\,i + y^2zx\,j + z^2xy\,k\right)$$
$$= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(y^2zx) + \frac{\partial}{\partial z}(z^2xy)$$
$$= 2xyz + 2xyz + 2xyz = 6xyz \quad \therefore \quad \text{div } \vec{A} = 6xyz$$

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & y^2zx & z^2xy \end{vmatrix}$$
$$= i(z^2x - y^2x) - j(z^2y - x^2y) + k(y^2z - x^2z)$$

$$\therefore \quad \text{curl } \vec{A} = x(z^2 - y^2)\,i + y(x^2 - z^2)\,j + z(y^2 - x^2)\,k$$

$$\nabla^2\vec{A} = \frac{\partial^2\vec{A}}{\partial x^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \frac{\partial^2\vec{A}}{\partial z^2}$$
$$= \frac{\partial}{\partial x}(2xyz\,i + y^2z\,j + z^2y\,k) + \frac{\partial}{\partial y}(x^2z\,i + 2xyz\,j + z^2x\,k) + \frac{\partial}{\partial z}(x^2y\,i + y^2x\,j + 2xyz\,k)$$
$$= 2yz\,i + 2xz\,j + 2xy\,k \quad \therefore \quad \nabla^2\vec{A} = 2\,(yzi + zxj + xyk)$$

Proof : Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ on the surface $\phi(x, y, z) = c$. Also let

$$\vec{r} = x(t)\,i + y(t)\,j + z(t)\,k$$

$\therefore \quad \dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt} i + \dfrac{dy}{dt} j + \dfrac{dz}{dt} k$ is tangential to the surface at P

We have $\quad \nabla\phi = \dfrac{\partial\phi}{\partial x} i + \dfrac{\partial\phi}{\partial y} j + \dfrac{\partial\phi}{\partial z} k$

Taking the dot product of these two vectors we have

$$\nabla\phi \cdot \frac{d\vec{r}}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} \quad \ldots(1)$$

Also, let us consider $\phi\ (x, y, z) = c$ where $x = x(t),\ y = y(t),\ z = z(t)$ and c is a constant.

Differentiating *w.r.t* t on both sides we have $\frac{d\phi}{dt} = 0$ and using the concept of the differentiation of composite functions (*Total derivative*) in L.H.S we obtain

$$\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} = 0$$

or $\nabla\phi \cdot \frac{d\vec{r}}{dt} = 0$ by using (1)

$\Rightarrow$ $\nabla\phi$ is perpendicular to $\frac{d\vec{r}}{dt}$

Since $\frac{d\vec{r}}{dt}$ is a vector tangential to the surface at P **we can conclude that $\nabla\phi$ is along the normal to the surface $\phi\ (x, y, z) = c$ at P.**

This proves the theorem.

Note : 1. *Obviously the unit vector normal $\hat{n}$ along $\nabla\phi$ is given by $\hat{n} = \nabla\phi / |\nabla\phi|$*

2. *The angle between the two surfaces is defined to be equal to the angle between their normals. If $\phi_1\ (x, y, z) = c_1$ and $\phi_2\ (x, y, z) = c_2$ be the equations of the two surfaces then*

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\ |\nabla\phi_2|}, \text{ where } \theta \text{ is the angle between the normals.}$$

If $\theta = \pi/2$ then the surfaces are said to intersect each other orthogonally.

When $\theta = \pi/2,\ \cos\theta = \cos(\pi/2) = 0 \Rightarrow \nabla\phi_1 \cdot \nabla\phi_2 = 0$

This is the condition for the surfaces to intersect at right angles.

If $\phi\ (x, y, z)$ is a scalar function and $\vec{d}$ is a given direction then $\nabla\phi \cdot \hat{n}$ where $\hat{n} = \vec{d} / |\vec{d}|$ is called as the *directional derivative* of ϕ along $\hat{n}$.

Proof : We have by definition $\nabla\phi \cdot \hat{n}$ is the directional derivative of ϕ along $\hat{n}$

Now, by the definition of the dot product we have

$$\nabla\phi \cdot \hat{n} = |\nabla\phi|\ |\hat{n}| \cos\theta$$

where θ is the angle between $\nabla\phi$ and $\hat{n}$. Since $|\hat{n}| = 1$ we have,

$$\nabla\phi \cdot \hat{n} = |\nabla\phi| \cos\theta$$

$\cos\theta$ when $\theta = 0$ has the maximum value equal to 1. If $\theta = 0$, $\nabla\phi$ coincides with $\hat{n}$ or we can say that $\nabla\phi$ will be along $\hat{n}$.

$\therefore$ **the directional derivative is maximum along $\nabla\phi$ and its maximum value is equal to $|\nabla\phi|$.**

This proves the theorem.

Note : *Maximum directional derivative of a scalar function ϕ at a point P is also called as the normal derivative of the scalar function at P. Normal derivative $= |\nabla\phi|$ at P.*

If $\phi(x, y, z) = c$ be the equation of a surface and $P(x_1, y_1, z_1)$ is a point on it then the equation of the ***tangent plane*** at the point P is given by

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

Also the equation of the ***normal line*** is given by

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C} \quad \text{where we have}$$

$$A = \left(\frac{\partial\phi}{\partial x}\right)_{(x_1, y_1, z_1)}, \quad B = \left(\frac{\partial\phi}{\partial y}\right)_{(x_1, y_1, z_1)}, \quad C = \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)}$$

If $\vec{V}(x, y, z)$ represents any physical quantity, the divergence of $\vec{V}$ gives the rate at which the physical quantity is originating at that point per unit volume.

An Illustration : Let us suppose that a fluid is moving such that its velocity at any point $P(x, y, z)$ is given by the vector point function $\vec{V}(x, y, z)$. Consider a small parallelopiped of volume $\delta x\ \delta y\ \delta z$ through which the fluid is passing.

If $\vec{V}(x, y, z) = v_1 i + v_2 j + v_3 k$ then $\text{div}\ \vec{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$ gives the total gain in the volume of the fluid per unit volume per unit time.

$\text{div}\ \vec{V} = 0$ is called as the continuity equation of an incompressible fluid.

A vector $\vec{V}$ whose divergence is zero is called a "solenoidal vector".

Curl means rotation. **A vector function $\vec{V}(x, y, z)$ is said to be "irrotational" if curl $\vec{V} = \vec{0}$**

An illustration : Let us suppose that a rigid body is rotating about a fixed axis through a point O. If $\vec{\omega}$ is the constant angular velocity and $\vec{v}$ is the velocity of a particle at a point $P(x, y, z)$ of the body having the position vector $\vec{r}$ then we know that $\vec{v} = \vec{\omega} \times \vec{r}$. We can easily show that curl $\vec{v} = 2\vec{\omega}$ *(Refer Example - 26)*

Thus curl of the velocity vector is equal to twice the angular velocity of rotation. This is an illustration to show that "curl" is analogous to "rotation". In general we can as well say that the curl of any vector point function will give the measure of the angular velocity at any point.

1. $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \Sigma \frac{\partial}{\partial x} i$

   $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Sigma \frac{\partial^2}{\partial x^2}$

2. grad $\phi = \nabla\phi$; div $\vec{A} = \nabla \cdot \vec{A}$; curl $\vec{A} = \nabla \times \vec{A}$

   Laplacian of $\phi = \nabla^2\phi = \nabla \cdot \nabla\phi$

3. $\nabla\phi$ is a vector normal to the surface $\phi(x, y, z) = c$ and $\nabla\phi / |\nabla\phi|$ is the unit vector normal to the surface.

4. Directional derivative of $\phi(x, y, z)$ along a given direction $\vec{D}$ is $\nabla\phi \cdot \hat{n}$ where $\hat{n} = \vec{D} / |\vec{D}|$ and also directional derivative is maximum along $\nabla\phi$

5. If $\phi(x, y, z)$ represents the temperature function then the directional derivative of ϕ along $\vec{D}$ is the rate of change of temperature along $\vec{D}$

1. Let $\phi = x^2 y + 2xz$. $\nabla\phi$ is a vector normal to the surface.

We have $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

ie., $\nabla\phi = (2xy + 2z) i + x^2 j + 2x k$

$[\nabla\phi]_{(2, -2, 3)} = -2i + 4j + 4k = 2(-i + 2j + 2k)$

The required unit vector normal $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Thus $\hat{n} = \dfrac{2(-i+2j+2k)}{\sqrt{2^2(1+4+4)}} = \dfrac{-i+2j+2k}{3}$

2. Let $\phi = xy^3z^2$. $\nabla\phi$ is a vector normal to the surface.

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

ie., $\nabla\phi = y^3z^2 i + 3xy^2z^2 j + 2xy^3 z k$

$$\left[\nabla\phi\right]_{(-1,\,-1,\,2)} = -4i - 12j + 4k = -4(i+3j-k)$$

The required unit vector normal $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Thus $\hat{n} = \dfrac{-4(i+3j-k)}{\sqrt{4^2(1+9+1)}} = -\dfrac{(i+3j-k)}{\sqrt{11}}$

3. Let $\phi = x^2y - 2xz + 2y^2z^4$

$\nabla\phi$ is a vector normal to the surface.

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

$$\nabla\phi = (2xy - 2z)i + (x^2 + 4yz^4)j + (-2x + 8y^2z^3)k$$

$$[\nabla\phi]_{(2,\,1,\,-1)} = 6i + 8j - 12k = 2(3i + 4j - 6k)$$

The unit vector normal $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|}$

Thus $\hat{n} = \dfrac{2(3i+4j-6k)}{\sqrt{2^2(9+16+36)}} = \dfrac{3i+4j-6k}{\sqrt{61}}$

4. $\phi = x^2yz + 4xz^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie., $$\nabla\phi = (2xyz + 4z^2)\, i + (x^2 z)\, j + (x^2 y + 8xz)\, k$$

$$[\nabla\phi]_{(1,\,-2,\,-1)} = 8i - j - 10k$$

The unit vector in the direction of $2i - j - 2k$ is

$$\hat{n} = \frac{2i - j - 2k}{\sqrt{4+1+4}} = \frac{2i - j - 2k}{3}$$

$\therefore$ the required directional derivative is

$$\nabla\phi \cdot \hat{n} = (8i - j - 10k) \cdot \frac{(2i - j - 2k)}{3}$$

Thus $$\nabla\phi \cdot \hat{n} = \frac{(8)(2) + (-1)(-1) + (-10)(-2)}{3} = \frac{37}{3}$$

5. $\phi = 4xz^3 - 3x^2y^2z$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie., $$\nabla\phi = (4z^3 - 6xy^2 z)\, i - (6x^2 yz)\, j + (12xz^2 - 3x^2y^2)\, k$$

$$[\nabla\phi]_{(2,\,-1,\,2)} = 8i + 48j + 84k$$

The unit vector in the direction of $2i - 3j + 6k$ is

$$\hat{n} = \frac{2i - 3j + 6k}{\sqrt{4+9+36}} = \frac{2i - 3j + 6k}{7}$$

$\therefore$ the required directional derivative is

$$\nabla\phi \cdot \hat{n} = (8i + 48j + 84k) \cdot \frac{2i - 3j + 6k}{7}$$

Thus $$\nabla\phi \cdot \hat{n} = \frac{(8)(2) + (48)(-3) + (84)(6)}{7} = \frac{376}{7}$$

6. $\phi = \dfrac{xz}{x^2 + y^2}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie., $\nabla\phi = z\left\{\dfrac{(x^2+y^2)\,1 - x\,(2x)}{(x^2+y^2)^2}\right\} i + \left\{\dfrac{-2xyz}{(x^2+y^2)^2}\right\} j + \left(\dfrac{x}{x^2+y^2}\right) k$

ie., $\nabla\phi = \dfrac{z\,(y^2-x^2)}{(x^2+y^2)^2}\, i - \dfrac{2xyz}{(x^2+y^2)^2}\, j + \dfrac{x}{x^2+y^2}\, k$

$$[\nabla\phi]_{(1,\,-1,\,1)} = \frac{1}{2} j + \frac{1}{2} k = \frac{1}{2}(j+k)$$

The unit vector normal in the direction of $\vec{A} = i - 2j + k$ is

$$\hat{n} = \frac{i-2j+k}{\sqrt{1+4+1}} = \frac{i-2j+k}{\sqrt{6}}$$

Thus the required directional derivative is

$$\nabla\phi \cdot \hat{n} = \frac{1}{2}(j+k) \cdot \frac{(i-2j+k)}{\sqrt{6}} = \frac{0-2+1}{2\sqrt{6}} = \frac{-1}{2\sqrt{6}}$$

7. $f = x^2 y^2 z^2$

$\therefore$ $\nabla f = \dfrac{\partial f}{\partial x} i + \dfrac{\partial f}{\partial y} j + \dfrac{\partial f}{\partial z} k$

ie., $\nabla f = 2xy^2 z^2\, i + 2x^2 y z^2\, j + 2x^2 y^2 z\, k$

$$[\nabla f]_{(1,\,1,\,-1)} = 2i + 2j - 2k = 2\,(i + j - k) \qquad \ldots (1)$$

In order to find the direction of the tangent, let $\vec{r} = xi + yj + zk$

ie., $\vec{r} = e^t\, i + (1 + 2\sin t)\, j + (t - \cos t)\, k$

$\dfrac{d\vec{r}}{dt} = e^t\, i + 2\cos t\, j + (1 + \sin t)\, k$ is the tangent vector.

We have $P = (x, y, z) = (1, 1, -1)$ by data.

$\therefore$ $e^t = 1$; $1 + 2\sin t = 1$; $t - \cos t = -1$

Here $e^t = 1 \Rightarrow t = 0$ and this value of t satisfy all the equations.

Thus $\left(\dfrac{d\vec{r}}{dt}\right)_{t=0} = i + 2j + k$ is the direction of the tangent and the unit vector in this direction is

$$\hat{n} = \frac{i+2j+k}{\sqrt{1+4+1}} = \frac{i+2j+k}{\sqrt{6}}$$

$\therefore$ the required directional derivative of $f(x, y, z)$ along the tangent to the given curve is

$$\nabla f \cdot \hat{n} = 2\,(i + j - k) \cdot \frac{(i + 2j + k)}{\sqrt{6}} \quad \text{by using (1)}$$

Thus $\nabla f \cdot \hat{n} = \frac{2}{\sqrt{6}}(1 + 2 - 1) = \frac{4}{\sqrt{6}} = 2\sqrt{2/3}$

>> Let $\phi = xyz$ so that we have,

$$\nabla \phi = \frac{\partial \phi}{\partial x}\, i + \frac{\partial \phi}{\partial y}\, j + \frac{\partial \phi}{\partial z}\, k = yz\,i + zx\,j + xy\,k$$

$$[\nabla \phi]_{(1,\,1,\,1)} = i + j + k$$

Let $\psi = x y^2 + y z^2 + z x^2$ and we have

$$\nabla \psi = \frac{\partial \psi}{\partial x}\, i + \frac{\partial \psi}{\partial y}\, j + \frac{\partial \psi}{\partial z}\, k$$

ie., $\nabla \psi = (y^2 + 2xz)\, i + (2xy + z^2)\, j + (2yz + x^2)\, k$

$[\nabla \psi]_{(1,1,1)} = 3i + 3j + 3k = 3\,(i + j + k)$ is the normal to the given surface at $(1, 1, 1)$

The unit vector along $3\,(i + j + k)$ is

$$\hat{n} = \frac{3\,(i + j + k)}{\sqrt{3^2\,(1 + 1 + 1)}} = \frac{i + j + k}{\sqrt{3}}$$

$\therefore$ the required directional derivative of ϕ along the normal to the given surface is

$$\nabla \phi \cdot \hat{n} = (i + j + k) \cdot \frac{(i + j + k)}{\sqrt{3}} = \frac{1 + 1 + 1}{\sqrt{3}} = \sqrt{3}$$

Also the equation of the tangent plane is

$$A\,(x - x_1) + B\,(y - y_1) + C\,(z - z_1) = 0$$

and the equation of the normal line is

$$(x - x_1)/A = (y - y_1)/B = (z - z_1)/C$$

where $(x_1, y_1\ z_1) = (1, 1, 1)$ and $A = \frac{\partial \psi}{\partial x},\ B = \frac{\partial \psi}{\partial y},\ C = \frac{\partial \psi}{\partial z}$ at $(1, 1, 1)$.

Thus we have the equation of the tangent plane :

$$3\,(x - 1) + 3\,(y - 1) + 3\,(z - 1) = 0 \quad \text{or} \quad x + y + z = 3.$$

The equation of the normal line is

$$\frac{x-1}{3} = \frac{y-1}{3} = \frac{z-1}{3} \quad \text{or} \quad x-1 = y-1 = z-1$$

>> We know that the directional derivative is maximum along the normal vector which being $\nabla\phi$.

Let $\quad \phi = x^2 y z^3$ so that we have

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k = 2xyz^3\, i + x^2 z^3\, j + 3x^2 yz^2\, k$$

$[\nabla\phi]_{(2,\,1,\,-1)} = -4\,i - 4\,j + 12\,k$ which is the required direction in which the directional derivative is maximum. The magnitude of this is given by $\sqrt{4^2(1+1+9)} = 4\sqrt{11}$

>> Let $\phi = x\sin z - y\cos z$ be the temperature field.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$$

ie., $\quad \nabla\phi = \sin z\, i - \cos z\, j + (x\cos z + y\sin z)\, k$

$[\nabla\phi]_{(0,\,0,\,0)} = -j$ **is the required direction** which means to say that the person should move down along the y-axis to get warm quickly.

11. *If the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(-1, 1, 2)$ has a maximum magnitude of 32 units in the direction parallel to y-axis, find a, b, c.*

>> Maximum directional derivative is along $\nabla\phi$ and in the direction parallel to y-axis the magnitude is given to be 32 units.

$\therefore \quad \nabla\phi \cdot j = 32$ at $(-1, 1, 2)$ ... (1)

We have $\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k$

ie., $\quad \nabla\phi = (ay^2 + 3cx^2z^2)\, i + (2axy + bz)\, j + (by + 2cx^3 z)\, k$

$$[\nabla\phi]_{(-1,\,1,\,2)} = (a + 12c)\, i + (-2a + 2b)\, j + (b - 4c)\, k$$

Now $\quad \nabla\phi \cdot j = -2a + 2b = 32$ by using (1), or $-a + b = 16$

Also since $\nabla\phi$ is parallel to the y – axis we must have
$a + 12c = 0$ and $b - 4c = 0$

Thus by solving the three equations :

$$-a + b = 16,\ a + 12c = 0,\ b - 4c = 0 \text{ we obtain}$$

$$\mathbf{a = -12,\quad b = 4,\quad c = 1}$$

>> The angle between the surfaces is defined to be equal to the angle between their normals and we know that $\nabla\phi$ is a vector normal to the surface. We have the equation of the two surfaces given by

$$x^2 + y^2 + z^2 = 9 \text{ and } x^2 + y^2 - z = 3$$

Let $\phi_1 = x^2 + y^2 + z^2$ and $\phi_2 = x^2 + y^2 - z$

We have $\nabla\phi = \dfrac{\partial\phi}{\partial x}\, i + \dfrac{\partial\phi}{\partial y}\, j + \dfrac{\partial\phi}{\partial z}\, k$

$\therefore \quad \nabla\phi_1 = 2x\,i + 2y\,j + 2zk$ and $\nabla\phi_2 = 2x\,i + 2y\,j - k$

$\therefore \quad [\nabla\phi_1]_{(2,\,-1,\,2)} = 4\,i - 2\,j + 4k = 2\,(2\,i - j + 2\,k)$

$\quad [\nabla\phi_2]_{(2,\,-1,\,2)} = 4i - 2j - k$

If θ is the angle between these two normals we have

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\;|\nabla\phi_2|}$$

ie., $$\cos\theta = \frac{2\,(8 + 2 - 2)}{\sqrt{2^2\,(4 + 1 + 4)}\;\sqrt{16 + 4 + 1}} = \frac{8}{3\sqrt{21}}$$

Thus $\mathbf{\theta = \cos^{-1}\,(8/3\sqrt{21})}$

>> Let $\phi = xy - z^2$ and we know that $\nabla\phi$ is a vector normal to the surface.

$$\nabla\phi = \frac{\partial\phi}{\partial x}\, i + \frac{\partial\phi}{\partial y}\, j + \frac{\partial\phi}{\partial z}\, k = y\,i + x\,j - 2z\,k$$

$\therefore \quad [\nabla\phi]_{(4,\,1,\,2)} = i + 4j - 4k \cdots \vec{A}$ (*say*)

$$[\nabla\phi]_{(3,\,3,\,-3)} = 3i+3j+6k = 3(i+j+2k) \cdot\cdot \vec{B} \quad (say)$$

If θ is the angle between the vectors $\vec{A}$ and $\vec{B}$ we have

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$$

Now, $$\cos\theta = \frac{3(1+4-8)}{\sqrt{1+16+16}\,\sqrt{3^2(1+1+4)}}$$

$$= \frac{-3}{\sqrt{33}\,\sqrt{6}} = \frac{-3}{\sqrt{3}\,\sqrt{11}\,\sqrt{3}\,\sqrt{2}}$$

Thus $\cos\theta = -\dfrac{1}{\sqrt{22}}$ or $\theta = \pi \pm \cos^{-1}\left(\dfrac{1}{\sqrt{22}}\right)$

>> First we have to ensure that the given point lies on both the surfaces.

Substituting $(1,\,-1,\,2)$ onto the equation $ax^2 - byz = (a+2)x$ we obtain

$$a+2b = a+2 \Rightarrow 2b = 2 \text{ or } b = 1$$

Also if $(1,\,-1,\,2)$ is substituted onto the L.H.S of the equation $4x^2y + z^3 = 4$ we get 4 which is equal to the R.H.S

$\Rightarrow$ the given point lies on both the surfaces when $b = 1$.

In order to find a we have to use the orthogonality condition $\nabla\phi_1 \cdot \nabla\phi_2 = 0$ where

$$\phi_1 = ax^2 - byz - (a+2)x \quad \text{and} \quad \phi_2 = 4x^2y + z^3$$

Now $\nabla\phi_1 = \{2ax - (a+2)\}i + (-bz)j + (-by)k$

$$\nabla\phi_2 = 8xyi + 4x^2j + 3z^2k$$

$\therefore$ $[\nabla\phi_1]_{(1,\,-1,\,2)} = (a-2)i - 2bj + bk$

$$[\nabla\phi_2]_{(1,\,-1,\,2)} = -8i + 4j + 12k$$

$\nabla\phi_1 \cdot \nabla\phi_2 = 0$ gives $-8(a-2) - 8b + 12b = 0$

ie., $-8a + 4b + 16 = 0$. But $b = 1$ and hence we get $a = 5/2$

Thus $a = 5/2$ and $b = 1$ are the required values.

>> $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k \quad ; \phi = 2z - x^3 y$

$\therefore \quad \nabla\phi = -3x^2 y i - x^3 j + 2k$

$[\nabla\phi]_{(1,-1,1)} = 3i - j + 2k$

Now $[\vec{A}]_{(1,-1,1)} = 2i + 3j + k$

$\therefore \quad [\vec{A} \cdot \nabla\phi]_{(1,-1,1)} = (3)(2) + (-1)(3) + (2)(1) = 5$

Also $\vec{A} \times \nabla\phi = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix}$

ie., $\vec{A} \times \nabla\phi = i(6+1) - j(4-3) + k(-2-9) = \mathbf{7i - j - 11k}$

>> [**Note** : *If a straight line make angles* α, β, γ *with the coordinate axes then* $\cos\alpha$, $\cos\beta$, $\cos\gamma$ *are the direction cosines of the line and it satisfy the identity* $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. *The direction being* $\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$.]

(a) Consider $\phi = x^2 y z^3$

$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

ie., $\nabla\phi = (2xyz^3)\, i + (x^2 z^3)\, j + (3x^2 y z^2)\, k$

$[\nabla\phi]_{(1,1,1)} = 2i + j + 3k$

Since the directional line is making equal angles with the co ordinate axes we have

$\alpha = \beta = \gamma$ and hence $\cos\alpha = \cos\beta = \cos\gamma = a$ (*say*)

$\therefore$ the direction $\vec{d} = ai + aj + ak$ and the associated unit vector $\hat{n} = \vec{d} / |\vec{d}|$

ie., $\hat{n} = \frac{a(i+j+k)}{\sqrt{a^2(1+1+1)}} = \frac{i+j+k}{\sqrt{3}}$

Hence the required directional derivative is

$\nabla\phi \cdot \hat{n} = (2i + j + 3k) \cdot \frac{i+j+k}{\sqrt{3}} = \frac{6}{\sqrt{3}} = \mathbf{2\sqrt{3}}$

(b) Unit vector along the x – axis is $\hat{i}$ and hence the required directional derivative is $\nabla\phi \cdot \hat{i}$. Thus we get,

$$(2i + j + 3k) \cdot i = 2$$

>> The normal derivative at $(2, 1, 1) = |\nabla\phi|$ at $(2, 1, 1)$

$$\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

ie., $\nabla\phi = \log z\, i - 2y\, j + (x/z)\, k$ and $[\nabla\phi]_{(2,1,1)} = -2j + 2k$

Thus $|\nabla\phi| = 2\sqrt{2}$

>> (a) $\nabla(fg) = \Sigma \frac{\partial}{\partial x}(fg)\, i = \Sigma \left(f \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial x} \right) i$

$$= f\, \Sigma \frac{\partial g}{\partial x} i + g\, \Sigma \frac{\partial f}{\partial x} i$$

Thus $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(b) $\nabla\left(\frac{f}{g}\right) = \Sigma \frac{\partial}{\partial x}\left(\frac{f}{g}\right) i = \Sigma \frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{g^2} i$

$$= \frac{1}{g^2}\left\{ g\, \Sigma \frac{\partial f}{\partial x} i - f\, \Sigma \frac{\partial g}{\partial x} i \right\}$$

Thus $\nabla\left(\frac{f}{g}\right) = \frac{g\, \nabla f - f\, \nabla g}{g^2}$

>> Let $\phi = x^3 + y^3 + z^3 - 3xyz$

$$\vec{F} = \text{grad}\, \phi = \nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$$

$\therefore \quad \vec{F} = (3x^2 - 3yz)\, i + (3y^2 - 3xz)\, j + (3z^2 - 3xy)\, k$

Now $\quad \text{div } \vec{F} = \nabla \cdot \vec{F}$

ie., $\quad = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot \left\{ (3x^2 - 3yz)\, i + (3y^2 - 3xz)\, j + (3z^2 - 3xy)\, k \right\}$

$$= \frac{\partial}{\partial x}(3x^2 - 3yz) + \frac{\partial}{\partial y}(3y^2 - 3xz) + \frac{\partial}{\partial z}(3z^2 - 3xy)$$

Thus $\quad \text{div } \vec{F} = 6x + 6y + 6z = 6(x + y + z)$

Also $\quad \text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2 - 3yz) & (3y^2 - 3xz) & (3z^2 - 3xy) \end{vmatrix}$

$$= i \left\{ \frac{\partial}{\partial y}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3y^2 - 3xz) \right\}$$

$$- j \left\{ \frac{\partial}{\partial x}(3z^2 - 3xy) - \frac{\partial}{\partial z}(3x^2 - 3yz) \right\} + k \left\{ \frac{\partial}{\partial x}(3y^2 - 3xz) - \frac{\partial}{\partial y}(3x^2 - 3yz) \right\}$$

$$= i \left\{ -3x - (-3x) \right\} - j \left\{ -3y - (-3y) \right\} + k \left\{ -3z - (-3z) \right\}$$

ie., $\quad \text{curl } \vec{F} = \vec{0}$

Thus **div $\vec{F} = 6(x + y + z)$; curl $\vec{F} = \vec{0}$**

>> Let $\phi = x y^3 z^2$

$\therefore \quad \vec{F} = \nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = y^3 z^2 i + 3xy^2 z^2 j + 2xy^3 z k$

$\text{div } \vec{F} = \nabla \cdot \vec{F}$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot \left(y^3 z^2 i + 3xy^2 z^2 j + 2xy^3 z k \right)$$

$$= \frac{\partial}{\partial x}(y^3 z^2) + \frac{\partial}{\partial y}(3xy^2 z^2) + \frac{\partial}{\partial z}(2xy^3 z)$$

$$= 0 + 6xyz^2 + 2xy^3 = 2xy(3z^2 + y^2)$$

$\therefore \quad \text{div } \vec{F} \text{ at } (1, -1, 1) = -2(3 + 1) = -8$

Also, $\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3 z^2 & 3xy^2 z^2 & 2xy^3 z \end{vmatrix}$

$= i\,(6xy^2 z - 6xy^2 z) - j\,(2y^3 z - 2y^3 z) + k\,(3y^2 z^2 - 3y^2 z^2)$

Thus $\text{curl } \vec{F} = \vec{0}$

>> $\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right)$

$\cdot \left\{ (3x^2 y - z)\, i + (xz^3 + y^4)\, j - 2x^3 z^2 k \right\}$

$= \frac{\partial}{\partial x}(3x^2 y - z) + \frac{\partial}{\partial y}(xz^3 + y^4) + \frac{\partial}{\partial z}(-2x^3 z^2)$

$\nabla \cdot \vec{F} = 6xy + 4y^3 - 4x^3 z = \phi$ *(say)*

Now grad $(\text{div } \vec{F}) = \text{grad } \phi = \nabla\phi$

We have $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k$

ie., $\nabla\phi = (6y - 12x^2 z)\, i + (6x + 12y^2)\, j + (-4x^3)\, k$

Thus $[\nabla\phi]_{(2,\,-1,\,0)} = -6i + 24j - 32k$

>> $\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot \left(xz^3 i - 2x^2 yz j + 2yz^4 k \right)$

$= \frac{\partial}{\partial x}(xz^3) - \frac{\partial}{\partial y}(2x^2 yz) + \frac{\partial}{\partial z}(2yz^4)$

$= z^3 - 2x^2 z + 8yz^3$

$\therefore$ $\nabla \cdot \vec{A}$ or div $\vec{A}$ at $(1, -1, 1) = 1 - 2 - 8 = -9$

Next $\nabla \times \vec{A}$ or curl $\vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$

$$= i\,(2z^4 + 2x^2 y) - j\,(0 - 3xz^2) + k\,(-4xyz - 0)$$

$$= 2\,(x^2 y + z^4)\; i + 3xz^2 j - 4xyz\, k$$

$\therefore$ $\nabla \times \vec{A}$ at $(1, -1, 1) = 3j + 4k$

Now $\nabla \cdot (\nabla \times \vec{A})$ or div (curl $\vec{A}$)

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot \{ 2\,(x^2 y + z^4)\; i + 3xz^2 j - 4xyz\, k \}$$

$$= \frac{\partial}{\partial x} \{ 2\,(x^2 y + z^4) \} + \frac{\partial}{\partial y} (3xz^2) + \frac{\partial}{\partial z} (-4xyz)$$

$$= 4xy - 4xy = 0$$

Thus $\nabla \cdot \vec{A} = -9$ and $\nabla \cdot (\nabla \times \vec{A}) = 0$

, >> curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+1) & 1 & -x-y \end{vmatrix}$

$$= i\,(-1-0) - j\,(-1-0) + k\,(0-1)$$

ie., curl $\vec{F} = -i + j - k$

$\therefore$ $\vec{F} \cdot$ curl $\vec{F} = \{ (x+y+1)\; i + j - (x+y)\; k \} \cdot (-i + j - k)$

ie., $= (x+y+1)\,(-1) + (1)\,(1) + (x+y)\,(1)$

$$= -x - y - 1 + 1 + x + y = 0$$

Thus $\vec{F} \cdot$ **curl** $\vec{F} = 0 \Rightarrow \vec{F}$ **is perpendicular to curl** $\vec{F}$

$$\text{curl } \vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 z & z^2 y \end{vmatrix}$$

$$= i(z^2 - y^2) - j(0-0) + k(0-x)$$

$\therefore$ $$\text{curl } \vec{A} = (z^2 - y^2)\, i - x\, k$$

Now $$\text{curl}(\text{curl } \vec{A}) = \nabla \times (\nabla \times \vec{A})$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z^2 - y^2) & 0 & -x \end{vmatrix} = i(0-0) - j(-1-2z) + k(0+2y)$$

$\therefore$ $$\textbf{curl}(\textbf{curl } \vec{A}) = (1+2z)\, j + 2y\, k$$

>> Observing the symmetric nature in $\vec{V}$, we can write

$$\text{div } \vec{V} = \nabla \cdot V = \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \left(\Sigma \frac{xi}{\sqrt{x^2 + y^2 + z^2}} \right)$$

ie., $$= \Sigma \frac{\partial}{\partial x} \left[\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right]$$

$$= \Sigma \frac{\sqrt{x^2 + y^2 + z^2} - (x) \frac{1}{2\sqrt{x^2 + y^2 + z^2}} (2x)}{x^2 + y^2 + z^2}$$

$$= \Sigma \frac{(x^2 + y^2 + z^2) - x^2}{(x^2 + y^2 + z^2)^{3/2}} = \Sigma \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

ie., $$= \frac{(y^2 + z^2) + (z^2 + x^2) + (x^2 + y^2)}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{3/2}}$$

$\therefore \quad \text{div } \vec{V} = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix}$$

(**Note** : *We should not remove the common factor appearing in the third row of the determinant as differential operators are involved in the second row*)

$$\therefore \quad \text{curl } \vec{V} = \Sigma\, i \left\{ \frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}} \right) - \frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}} \right) \right\}$$

$$= \Sigma\, i \left\{ z \left(\frac{-1}{2} \right) (x^2+y^2+z^2)^{-3/2} (2y) - y \left(\frac{-1}{2} \right) (x^2+y^2+z^2)^{-3/2} (2z) \right\}$$

$$= \Sigma\, i \left\{ -yz (x^2+y^2+z^2)^{-3/2} + yz (x^2+y^2+z^2)^{-3/2} \right\} = \vec{0}$$

Thus $\quad \text{curl } \vec{V} = \vec{0}$

26. *If $\vec{V} = \vec{\omega} \times \vec{r}$ show that curl $\vec{V} = 2\vec{\omega}$ where $\vec{\omega}$ is a constant vector.*

>> Let $\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$ be the constant vector.

We have $\quad \vec{r} = x\,i + y\,j + z\,k$

$$\vec{V} = \vec{\omega} \times \vec{r} = \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \Sigma\, i\,(\omega_2 z - \omega_3 y)$$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & (\omega_3 x - \omega_1 z) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= \Sigma\, i \left(\omega_1 - (-\omega_1) \right) = \Sigma\, 2\omega_1 i = 2(\omega_1 i + \omega_2 j + \omega_3 k) = 2\vec{\omega}$$

$\therefore \quad \text{curl } \vec{V} = 2\vec{\omega} \quad \text{or} \quad \vec{\omega} = \frac{1}{2} \text{curl } \vec{V}$

Remark : *We had referred to this example while giving the physical meaning of "curl". The theoretical version of this problem is as follows :*

When a rigid body is in motion the angular velocity is equal to half the curl of its linear velocity at any point.

27. Let $\vec{A} = a_1 i + a_2 j + a_3 k \quad ; \quad \vec{r} = x i + y j + z k$

(Note : *This step is common for all the five examples*)

$$\vec{A} \cdot \vec{r} = \Sigma \, a_1 x$$

$$\therefore \quad \nabla (\vec{A} \cdot \vec{r}) = \left(\Sigma \frac{\partial}{\partial x} i \right) (\Sigma \, a_1 x) = \Sigma \, a_1 i = \vec{A}$$

Thus $\quad \nabla (\vec{A} \cdot \vec{r}) = \vec{A} \quad$ or $\quad \textbf{grad } (\vec{A} \cdot \vec{r}) = \vec{A}$

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28. $$\vec{A} \times \vec{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \Sigma \, i (a_2 z - a_3 y)$$

$$\therefore \quad \nabla \cdot (\vec{A} \times \vec{r}) = \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \Sigma \, i (a_2 z - a_3 y)$$

$$= \Sigma \frac{\partial}{\partial x} (a_2 z - a_3 y) = 0 + 0 + 0 = 0$$

Thus $\quad \nabla \cdot (\vec{A} \times \vec{r}) = 0 \quad$ or $\quad \textbf{div } (\vec{A} \times \vec{r}) = 0$

---

29. This example is same as Example-26 as we have $\vec{A}$ in place of $\vec{\omega}$ being a constant vector. We have proved in Example-26 that curl $(\vec{\omega} \times \vec{r}) = 2\vec{\omega}$

Thus $\quad \textbf{curl } (\vec{A} \times \vec{r}) = 2\vec{A} \quad$ or $\quad \nabla \times (\vec{A} \times \vec{r}) = 2\vec{A}$

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30. $\vec{A} \cdot \vec{r} = a_1 x + a_2 y + a_3 z$

Also $(\vec{A} \cdot \vec{r})\, \vec{r} = (a_1 x + a_2 y + a_3 z)\,(x i + y j + z k)$

ie., $(\vec{A} \cdot \vec{r})\, \vec{r} = \Sigma\,(a_1 x^2 + a_2 xy + a_3 xz)\, i$

Now curl $\{(\vec{A} \cdot \vec{r})\, \vec{r}\} = \nabla \times \{(\vec{A} \cdot \vec{r})\, \vec{r}\}$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (a_1 x^2 + a_2 xy + a_3 xz) & (a_2 y^2 + a_3 yz + a_1 xy) & (a_3 z^2 + a_1 zx + a_2 yz) \end{vmatrix}$$

$$= \Sigma i\,(a_2 z - a_3 y) \qquad \ldots (1)$$

Now consider R.H.S $= \vec{A} \times \vec{r}$

ie., $$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \Sigma\,(a_2 z - a_3 y) \qquad \ldots (2)$$

Comparing (1) and (2) we have,

$$\textbf{curl}\, \{(\vec{A} \cdot \vec{r})\, \vec{r}\} = \vec{A} \times \vec{r} \quad \text{or} \quad \nabla \times \{(\vec{A} \cdot \vec{r})\, \vec{r}\} = \vec{A} \times \vec{r}$$

31. $$\vec{r} \times \vec{A} = \begin{vmatrix} i & j & k \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix} = \Sigma\, i\,(a_3 y - a_2 z)$$

$$\therefore \quad \vec{A} \times (\vec{r} \times \vec{A}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ (a_3 y - a_2 z) & (a_1 z - a_3 x) & (a_2 x - a_1 y) \end{vmatrix}$$

$$= \Sigma\, i\,(a_2^2 x - a_1 a_2 y - a_1 a_3 z + a_3^2 x)$$

Now $\nabla \cdot \{\vec{A} \times (\vec{r} \times \vec{A})\}$

$$= \left(\Sigma\, \frac{\partial}{\partial x} i\right) \cdot \Sigma\, i\left(a_2^2 x - a_1 a_2 y - a_1 a_3 z + a_3^2 x\right)$$

$$= \Sigma\, \frac{\partial}{\partial x}\,(a_2^2 x - a_1 a_2 y - a_1 a_3 z + a_3^2 x)$$

$$= \Sigma\ (a_2^2 + a_3^2) = (a_2^2 + a_3^2) + (a_3^2 + a_1^2) + (a_1^2 + a_2^2)$$

$$= 2\,(a_1^2 + a_2^2 + a_3^2) = 2\ |\vec{A}\,|^2$$

$$\because \qquad |\vec{A}\,| = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad \Rightarrow \quad |\vec{A}\,|^2 = a_1^2 + a_2^2 + a_3^2$$

Thus $\quad$ **div** $\{\vec{A} \times (\vec{r} \times \vec{A}\,)\} = 2\ |\vec{A}\,|^2$

Remark : *In all these five examples we must observe the symmetry and use "sigma" notation to arrive at the desired result quickly.*

Note : *These are also a set of problems based on the aspect of symmetry and the problems are worked in the general form involving "n". Particular cases for n can also be asked. The notation and meaning concerning $\vec{r}$ and r are standard which need not be explicitly mentioned while asking these problems or particular cases of these.*

32. $r = |\,\vec{r}\,| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow \qquad r^2 = x^2 + y^2 + z^2$ and differentiating partially *w.r.t* x we get

$$2r\,\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{Also} \quad \frac{\partial r}{\partial y} = \frac{y}{r}\,, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Note : *The expression for these three partial derivatives will be of use in all the problems and the same will not be worked in every problem.*

Now $\quad \nabla\,(r^n) = \left(\Sigma\ \frac{\partial}{\partial x}\ i\right)(r^n)$

$$= \Sigma\ n\,r^{n-1}\,\frac{\partial r}{\partial x}\ i = \Sigma\ n\,r^{n-1}\left(\frac{x}{r}\right) i$$

$$= \Sigma\ n\,r^{n-2}\,x\,i = n\,r^{n-2}\ \Sigma\ x\,i = n\,r^{n-2}\ \vec{r}$$

Thus $\quad \nabla\,(r^n) =$ **grad** $(r^n) = n\,r^{n-2}\ \vec{r}$

33. $r^n \vec{r} = r^n \Sigma xi = \Sigma (r^n x) i$

$$\therefore \quad \nabla \cdot (r^n \vec{r}) = \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \Sigma (r^n x) i$$

$$= \Sigma \frac{\partial}{\partial x} (r^n x) = \Sigma \left(r^n + n r^{n-1} \frac{\partial r}{\partial x} x \right)$$

ie.,
$$= \Sigma \left(r^n + n r^{n-1} \frac{x}{r} x \right) = \Sigma \left(r^n + n r^{n-2} x^2 \right)$$

On expanding the summation we get,

$$(r^n + n r^{n-2} x^2) + (r^n + n r^{n-2} y^2) + (r^n + n r^{n-2} z^2)$$

$$= 3 r^n + n r^{n-2} (x^2 + y^2 + z^2)$$

$$= 3r^n + n r^{n-2} r^2 = 3 r^n + n r^n = (n+3) r^n$$

Thus $\quad \nabla \cdot (r^n \vec{r}) = \textbf{div} (r^n \vec{r}) = (n+3) r^n$

34. $r^n \vec{r} = r^n \Sigma x i = \Sigma (r^n x) i$

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \Sigma i \left\{ \frac{\partial}{\partial y} (r^n z) - \frac{\partial}{\partial z} (r^n y) \right\}$$

$$= \Sigma i \left\{ n r^{n-1} \frac{\partial r}{\partial y} z - n r^{n-1} \frac{\partial r}{\partial z} y \right\}$$

$$= \Sigma i \left\{ n r^{n-1} \frac{y}{r} z - n r^{n-1} \frac{z}{r} y \right\}$$

$$= \Sigma i (n r^{n-2} yz - n r^{n-2} yz) = \vec{0}$$

Thus $\quad \nabla \times (r^n \vec{r}) = \textbf{curl} (r^n \vec{r}) = \vec{0}$

35. $\nabla^2(r^n) = \Sigma \frac{\partial^2}{\partial x^2}(r^n) = \Sigma \frac{\partial}{\partial x}\frac{\partial}{\partial x}(r^n)$

$$= \Sigma \frac{\partial}{\partial x}\left(n\, r^{n-1}\frac{\partial r}{\partial x}\right) = \Sigma \frac{\partial}{\partial x}\left\{n\, r^{n-1}\left(\frac{x}{r}\right)\right\}$$

ie., $$= \Sigma \frac{\partial}{\partial x}\left(n\, r^{n-2} x\right) = n\, \Sigma \left(r^{n-2} + (n-2)\, r^{n-3}\frac{\partial r}{\partial x} x\right)$$

ie., $$= n\, \Sigma \left(r^{n-2} + (n-2)\, r^{n-3}\frac{x}{r} x\right) = n\, \Sigma \left(r^{n-2} + (n-2)\, r^{n-4} x^2\right)$$

On expanding the summation we get,

$$= n\left\{\left(r^{n-2} + (n-2)\, r^{n-4} x^2\right) + \left(r^{n-2} + (n-2)\, r^{n-4} y^2\right) + \left(r^{n-2} + (n-2)\, r^{n-4} z^2\right)\right\}$$

$$= n\left\{3\, r^{n-2} + (n-2)\, r^{n-4}(x^2+y^2+z^2)\right\}$$

$$= n\left\{3\, r^{n-2} + (n-2)\, r^{n-4} r^2\right\}$$

$$= n\left\{3\, r^{n-2} + (n-2)\, r^{n-2}\right\} = n r^{n-2}(3+n-2)$$

$$= n\, r^{n-2}(n+1)$$

Thus $\boldsymbol{\nabla^2(r^n) = n(n+1)\, r^{n-2}}$

Note : *A few important particular cases of these are given and is left as an exercise for the reader to work out these independently.*

Prove with usual meanings the following :

1. $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$ 2. $\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = 0$ 3. $\nabla^2\left(\frac{1}{r}\right) = 0$

>> $\nabla^2[f(r)] = \Sigma \frac{\partial^2}{\partial x^2}[f(r)] = \Sigma \frac{\partial}{\partial x}\frac{\partial}{\partial x}[f(r)]$

ie., $$= \Sigma \frac{\partial}{\partial x}\left\{f'(r)\frac{\partial r}{\partial x}\right\} = \Sigma \frac{\partial}{\partial x}\left\{f'(r)\frac{x}{r}\right\}$$

$\because \quad r^2 = x^2+y^2+z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$

$$\nabla^2[f(r)] = \Sigma\left\{f'(r)\left[\frac{r - x\frac{\partial r}{\partial x}}{r^2}\right] + f''(r)\frac{\partial r}{\partial x}\frac{x}{r}\right\}$$

ie., $$= \Sigma f'(r)\left[\frac{r - x\left(\frac{x}{r}\right)}{r^2}\right] + \Sigma f''(r)\frac{x}{r}\frac{x}{r}$$

$$= \Sigma\frac{f'(r)}{r^3}(r^2 - x^2) + \Sigma f''(r)\frac{x^2}{r^2}$$

$$= \frac{f'(r)}{r^3}\left\{(r^2-x^2) + (r^2-y^2) + (r^2-z^2)\right\} + \frac{f''(r)}{r^2}\Sigma x^2$$

$$= \frac{f'(r)}{r^3}\left\{3r^2 - (x^2+y^2+z^2)\right\} + \frac{f''(r)}{r^2}r^2$$

$$= \frac{f'(r)}{r^3}(3r^2 - r^2) + f''(r) = \frac{2}{r}f'(r) + f''(r)$$

Thus $\nabla^2[f(r)] = \frac{2}{r}f'(r) + f''(r)$ ...(1)

Now let $f(r) = e^r \quad \therefore \; f'(r) = e^r = f''(r)$

Thus (1) becomes

$$\nabla^2(e^r) = \frac{2}{r}e^r + e^r = e^r\left(\frac{2}{r}+1\right)$$

Thus $\nabla^2(e^r) = e^r\left(\frac{2}{r}+1\right)$ ...(2)

Also $\nabla^4(e^r) = \nabla^2\,\nabla^2(e^r) = \nabla^2\left\{e^r\left(\frac{2}{r}+1\right)\right\}$

Now taking $f(r) = e^r\left(\frac{2}{r}+1\right)$

$$f'(r) = e^r\left(\frac{-2}{r^2}\right) + e^r\left(\frac{2}{r}+1\right) \qquad \text{...(3)}$$

$$f''(r) = e^r\left(\frac{4}{r^3}\right) + e^r\left(\frac{-2}{r^2}\right) + e^r\left(\frac{-2}{r^2}\right) + e^r\left(\frac{2}{r}+1\right)$$

ie., $f''(r) = \frac{4}{r^3}e^r - \frac{4}{r^2}e^r + \frac{2}{r}e^r + e^r$ . . . (4)

Substituting (3) and (4) in (1) we get

$$\nabla^2 \left\{ e^r \left(\frac{2}{r} + 1 \right) \right\}$$

$$= \frac{2}{r} \left\{ -\frac{2e^r}{r^2} + \frac{2e^r}{r} + e^r \right\} + \frac{4}{r^3}e^r - \frac{4}{r^2}e^r + \frac{2}{r}e^r + e^r$$

Simplifying R.H.S we obtain

$$\nabla^2 \left\{ \nabla^2 (e^r) \right\} = e^r \left(\frac{4}{r} + 1 \right)$$

Thus $\nabla^4 (e^r) = e^r \left(\frac{4}{r} + 1 \right)$

>> We need to first obtain $\nabla \cdot \left(\frac{\vec{r}}{r^2} \right)$ similar to Example - 33.

On doing so we can obtain

$$\nabla \cdot \left(\frac{\vec{r}}{r^2} \right) = \frac{1}{r^2} \quad \textit{(particular case where } n = -2)$$

Then $\nabla^2 \left(\frac{1}{r^2} \right)$ is to be worked out similar to Example-35

On doing so we can obtain, *(particular case where* $n = -2$) $\nabla^2 \left(\frac{1}{r^2} \right) = \frac{2}{r^4}$

Thus $\nabla^2 \left\{ \nabla \cdot \left(\frac{\vec{r}}{r^2} \right) \right\} = \frac{2}{r^4}$

>> We shall denote $r = |\vec{r}|$ so that $\vec{A} = \frac{\vec{r}}{r}$

div $\vec{A} = \nabla \cdot \left(\frac{\vec{r}}{r} \right)$ is similar to Example - 33.

On doing so we can obtain

$$\nabla \cdot \left(\frac{\vec{r}}{r}\right) = \frac{2}{r} \quad \text{(particular case where } n = -1)$$

$\therefore$ $$\text{grad (div } A) = \text{grad}\left(\frac{2}{r}\right) = 2\,\nabla\left(\frac{1}{r}\right)$$

$\nabla\left(\frac{1}{r}\right)$ is a particular case of Example - 32 for $n = -1$

We can obtain $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$

Thus **grad (div $\vec{A}$) = $\nabla(\nabla \cdot \vec{A}) = -\frac{2\vec{r}}{r^3}$**

>> $$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$$

(i) $\nabla \cdot \hat{r} = \nabla \cdot \left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$ *(Refer Example-38)*

(ii) $\nabla \times \hat{r} = \nabla \times \left(\frac{\vec{r}}{r}\right)$ is a particular case of Example - 34 for $n = -1$

We can obtain $\nabla \times \left(\frac{\vec{r}}{r}\right) = 0$ or $\nabla \times \hat{r} = \vec{0}$

Thus $\nabla \cdot \hat{r} = 2/r$ and $\nabla \times \hat{r} = \vec{0}$

Remark : *Infact we have worked this example without using the notation $\vec{r}$ and $|\vec{r}|$. The same results have been obtained in Example - 25 where $\vec{V}$ is nothing but $\hat{r}$.*

>> Consider $\phi = 2x^3 y^2 z^4$

$$\text{grad } \phi = \nabla\phi = \frac{\partial \phi}{\partial x}\, i + \frac{\partial \phi}{\partial y}\, j + \frac{\partial \phi}{\partial z}\, k$$

ie., $$\nabla\phi = (6x^2 y^2 z^4)\, i + (4x^3 y z^4)\, j + (8x^3 y^2 z^3)\, k$$

$$\text{div (grad } \phi) = \nabla \cdot \nabla\phi$$

ie., $= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot \left(6x^2 y^2 z^4 i + 4x^3 yz^4 j + 8x^3 y^2 z^3 k \right)$

$$= \frac{\partial}{\partial x} (6x^2 y^2 z^4) + \frac{\partial}{\partial y} (4x^3 yz^4) + \frac{\partial}{\partial z} (8x^3 y^2 z^3)$$

$$\therefore \quad \nabla \cdot (\nabla \phi) = 12 xy^2 z^4 + 4 x^3 z^4 + 24 x^3 y^2 z^2 \quad \ldots (1)$$

Next $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ ... (2)

Consider $\phi = 2x^3 y^2 z^4$

$$\frac{\partial \phi}{\partial x} = 6x^2 y^2 z^4, \; \frac{\partial \phi}{\partial y} = 4x^3 yz^4, \; \frac{\partial \phi}{\partial z} = 8x^3 y^2 z^3$$

$$\frac{\partial^2 \phi}{\partial x^2} = 12xy^2 z^4, \; \frac{\partial^2 \phi}{\partial y^2} = 4x^3 z^4, \; \frac{\partial^2 \phi}{\partial z^2} = 24x^3 y^2 z^2$$

Adding these results we have according to (2)

$$\nabla^2 \phi = 12xy^2 z^4 + 4x^3 z^4 + 24x^3 y^2 z^2 \quad \ldots (3)$$

Thus by comparing (1) and (3) $\nabla \cdot \nabla \phi = \nabla^2 \phi$ **is verified.**

We have already referred to these while giving the physical meaning of divergence and curl.

A vector field $\vec{F}$ is said to be **solenoidal if** div $\vec{F} = 0$ and **irrotational if** curl $\vec{F} = \vec{0}$.

Irrotational field is also called as *conservative field* or *potential field.*

When $\vec{F}$ is irrotational there always exists a scalar point function ϕ such that $\nabla \phi = \vec{F}$. Then ϕ is called a **scalar potential** of $\vec{F}$

>> div $\vec{F} = \nabla \cdot \vec{F}$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \right) \cdot \left(\frac{x}{x^2+y^2} i + \frac{y}{x^2+y^2} j \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} \right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2+y^2} \right)$$

$$= \left\{ \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} \right\} + \left\{ \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} \right\}$$

$$= \frac{1}{(x^2+y^2)^2} \left\{ y^2 - x^2 + x^2 - y^2 \right\} = 0$$

Thus div $\vec{F} = 0 \Rightarrow$ $\vec{F}$ **is solenoidal.**

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= 0i + 0j + k \left\{ \frac{\partial}{\partial x} \left(\frac{y}{x^2+y^2} \right) - \frac{\partial}{\partial y} \left(\frac{x}{x^2+y^2} \right) \right\}$$

$$= k \left(\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} \right) = \vec{0}$$

Thus curl $\vec{F} = \vec{0} \Rightarrow$ $\vec{F}$ **is irrotational.**

>> $\vec{F} = x^{a+b} y^b z^b i + x^b y^{a+b} z^b j + x^b y^b z^{a+b} k$

ie., $\vec{F} = f_1 i + f_2 j + f_3 k$ (*say*) where

$$f_1 = x^{a+b} y^b z^b,\ f_2 = x^b y^{a+b} z^b,\ f_3 = x^b y^b z^{a+b} \qquad \ldots (1)$$

Now div $\vec{F} = 0 \Rightarrow \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z} = 0$

ie., $(a+b)\, x^{a+b-1} y^b z^b + (a+b)\, x^b y^{a+b-1} z^b + (a+b)\, x^b y^b z^{a+b-1} = 0$

or $(a+b)\,(xyz)^b \left\{ x^{a-1} + y^{a-1} + z^{a-1} \right\} = 0$

This equation is identically satisfied only when $a + b = 0$

Thus $\vec{F}$ **is solenoidal if** $a + b = 0$

Next consider curl $\vec{F} = \vec{0}$

ie., $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \vec{0}$

ie., $\Sigma \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) i = \vec{0}$ and by using (1) we have

$$\Sigma \left(bx^b y^{b-1} z^{a+b} - bx^b y^{a+b} z^{b-1} \right) i = \vec{0}$$

ie., $\Sigma\, bx^b \left(y^{b-1} z^{a+b} - y^{a+b} z^{b-1} \right) i = \vec{0}$

Since $x^b \neq 0$, the equation is identically satisfied when $b = 0$ or $a + b = b - 1$

Thus $b = 0$ or $a = -1$ is the required condition for $\vec{F}$ **to be irrotational.**

If $\vec{F}$ is both solenoidal and irrotational we must have

$a + b = 0$ and $b = 0$ or $a + b = 0$ and $a = -1$.

ie., $a = 0$ **and** $b = 0$ **or** $a = -1$ **and** $b = 1$

Remark : If $a = 0$ *and* $b = 0$ *we have* $\vec{F} = i + j + k$

If $a = -1,\ b = 1$ *we have* $\vec{F} = yz\,i + zx\,j + xy\,k$

It can be easily seen that $div\ \vec{F} = 0$ *and* $curl\ \vec{F} = \vec{0}$ *in both the forms of* $\vec{F}$

>> We have to show that curl $\vec{F} = \vec{0}$

$\therefore$ curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y+z) & (z+x) & (x+y) \end{vmatrix}$

$$= i(1-1) - j(1-1) + k(1-1) = \vec{0}$$

Hence $\vec{F}$ **is irrotational.**

Now let us consider $\nabla\phi = \vec{F}$

ie., $\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (y+z)\, i + (z+x)\, j + (x+y)\, k$

$\Rightarrow \quad \dfrac{\partial \phi}{\partial x} = y + z \qquad \therefore \quad \phi = \int (y+z)\, dx + f_1(y, z)$

ie., $\quad \phi = xy + xz + f_1(y, z) \qquad \ldots (1)$

$\dfrac{\partial \phi}{\partial y} = z + x \qquad \therefore \quad \phi = \int (z+x)\, dy + f_2(x, z)$

ie., $\quad \phi = yz + xy + f_2(x, z) \qquad \ldots (2)$

$\dfrac{\partial \phi}{\partial z} = x + y \qquad \therefore \quad \phi = \int (x+y)\, dz + f_3(x, y)$

ie., $\quad \phi = xz + yz + f_3(x, y) \qquad \ldots (3)$

[*Now we need to suitably choose the arbitrary functions* $f_1(y, z)$, $f_2(x, z)$, $f_3(x, y)$ *such that we obtain an unique expression for* ϕ. *To choose* $f_1(y, z)$ *we look into the equations (2) and (3) and select terms which do not contain* x. *It can be terms with* y *or* z *or* y *and* z. *Similarly we have to choose* $f_2(x, z)$ *from (1) and (3),* $f_3(x, y)$ *from (1) and (2)*]

Let us choose $f_1(y, z) = yz$, $f_2(z, x) = xz$, $f_3(x, y) = xy$ from (1), (2) & (3).

Thus the required $\phi = xy + yz + zx$

>> We have to show that curl $\vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy^2 + yz) & (2x^2 y + xz + 2yz^2) & (2y^2 z + xy) \end{vmatrix}$$

$$= i(4yz + x - x - 4yz) - j(y - y) + k(4xy + z - 4xy - z) = \vec{0}$$

$\therefore \quad$ **$\vec{F}$ is conservative.**

Now we have to find ϕ such $\nabla \phi = \vec{F}$

ie., $\quad \dfrac{\partial \phi}{\partial x} i + \dfrac{\partial \phi}{\partial y} j + \dfrac{\partial \phi}{\partial z} k = (2xy^2 + yz)\, i + (2x^2 y + xz + 2yz^2)\, j + (2y^2 z + xy)\, k$

$\Rightarrow \quad \dfrac{\partial \phi}{\partial x} = 2xy^2 + yz \qquad \therefore \quad \phi = \int (2xy^2 + yz)\, dx + f_1(y, z)$

ie., $\quad \phi = x^2 y^2 + xyz + f_1(y, z) \qquad \ldots (1)$

$$\frac{\partial \phi}{\partial y} = 2x^2 y + xz + 2yz^2$$

$$\therefore \quad \phi = \int (2x^2 y + xz + 2yz^2)\, dy + f_2(x, z)$$

$$ie., \quad \phi = x^2 y^2 + xyz + y^2 z^2 + f_2(x, z) \quad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = 2y^2 z + xy \quad \therefore \quad \phi = \int (2y^2 z + xy)\, dz + f_3(x, y)$$

$$ie., \quad \phi = y^2 z^2 + xyz + f_3(x, y) \quad \ldots (3)$$

Let us choose $f_1(y, z) = y^2 z^2$, $f_2(x, z) = 0$, $f_3(x, y) = x^2 y^2$ from (1), (2) & (3).

Thus $\phi = x^2 y^2 + y^2 z^2 + xyz$ is the required scalar potenual.

>> We have to find 'a' such that curl $\vec{F} = \vec{0}$

$$ie., \quad \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^3) & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \vec{0}$$

$$ie., \quad i(0-0) - j\{(1-a)z^2 + 3z^2\} + k\{(a-2)2x - ax\} = \vec{0}$$

$$ie., \quad (a-4)z^2 j + (a-4)x k = \vec{0}$$

The above equation is identically satisfied when $a = 4$.

Now consider $\nabla\phi = (\vec{F})_{a=4}$

$$ie., \quad \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k = (4xy - z^3) i + 2x^2 j - 3xz^2 k$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = 4xy - z^3 \quad \therefore \quad \phi = \int (4xy - z^3)\, dx + f_1(y, z)$$

$$ie., \quad \phi = 2x^2 y - xz^3 + f_1(y, z) \quad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = 2x^2 \quad \therefore \quad \phi = \int 2x^2\, dy + f_2(x, z)$$

ie., $\phi = 2x^2 y + f_2(x, z)$ ...(2)

$$\frac{\partial \phi}{\partial z} = -3xz^2 \quad \therefore \quad \phi = \int -3xz^2\, dz + f_3(x, y)$$

ie., $\phi = -xz^3 + f_3(x, y)$ ...(3)

Let us choose $f_1(y, z) = 0$, $f_2(x, z) = -xz^3$, $f_3(x, y) = 2x^2 y$ from (1), (2) & (3).

Thus $\phi = 2x^2 y - xz^3$

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46. *Find constants a and b such that $\vec{F} = (axy + z^3)\, i + (3x^2 - z)\, j + (bxz^2 - y)\, k$ is irrotational. Also find a scalar function ϕ such that $\vec{F} = \nabla\phi$.*

\>> We have to find a and b such that curl $\vec{F} = \vec{0}$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (axy + z^3) & (3x^2 - z) & (bxz^2 - y) \end{vmatrix} = \vec{0}$$

ie., $i(-1+1) - j(bz^2 - 3z^2) + k(6x - ax) = \vec{0}$

ie., $-z^2(b-3)\, j + x(6-a)\, k = \vec{0}$

The above equation is identically satisfied when

$$b - 3 = 0 \text{ and } 6 - a = 0 \qquad \therefore \ \mathbf{a = 6 \text{ and } b = 3}$$

Now consider $\nabla\phi = \vec{F}$ when $a = 6,\ b = 3$

ie., $\dfrac{\partial \phi}{\partial x} i + \dfrac{\partial \phi}{\partial y} j + \dfrac{\partial \phi}{\partial z} k = (6xy + z^3)\, i + (3x^2 - z)\, j + (3xz^2 - y)\, k$

$\Rightarrow$ $\dfrac{\partial \phi}{\partial x} = 6xy + z^3 \quad \therefore \quad \phi = \int (6xy + z^3)\, dx + f_1(y, z)$

ie., $\phi = 3x^2 y + xz^3 + f_1(y, z)$ ...(1)

$$\frac{\partial \phi}{\partial y} = (3x^2 - z) \quad \therefore \quad \phi = \int (3x^2 - z)\, dy + f_2(x, z)$$

ie., $\phi = 3x^2 y - yz + f_2(x, z)$ ...(2)

$$\frac{\partial \phi}{\partial z} = (3xz^2 - y) \quad \therefore \quad \phi = \int (3xz^2 - y)\, dz + f_3(x, y)$$

ie., $\phi = xz^3 - yz + f_3(x, y)$ ...(3)

Let us choose $f_1(y, z) = -yz$, $f_2(x, z) = xz^3$, $f_3(x, y) = 3x^2 y$ from (1), (2) & (3).

Thus the required $\phi = 3x^2 y + xz^3 - yz$

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47 *If $\vec{F} = (x+y+az)\,i + (bx+2y-z)\,j + (x+cy+2z)\,k$ find a, b, c such that curl $\vec{F} = \vec{0}$ and then find ϕ such that $\vec{F} = \nabla\phi$.*

>> $$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x+y+az) & (bx+2y-z) & (x+cy+2z) \end{vmatrix} = \vec{0}$$

ie., $i(c+1) - j(1-a) + k(b-1) = \vec{0}$

$\Rightarrow$ $c+1 = 0,\ 1-a = 0,\ b-1 = 0$

$\therefore$ $a = 1,\ b = 1,\ c = -1$ are the required values.

Now consider $\nabla\phi = \vec{F}$ when $a = 1,\ b = 1,\ c = -1$.

ie., $$\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k = (x+y+z)\,i + (x+2y-z)\,j + (x-y+2z)\,k$$

$\Rightarrow$ $\dfrac{\partial\phi}{\partial x} = x+y+z \quad \therefore \quad \phi = \int (x+y+z)\,dx + f_1(y, z)$

ie., $\phi = \dfrac{x^2}{2} + xy + xz + f_1(y, z)$ ...(1)

$\dfrac{\partial\phi}{\partial y} = x+2y-z \quad \therefore \quad \phi = \int (x+2y-z)\,dy + f_2(x, z)$

ie., $\phi = xy + y^2 - yz + f_2(x, z)$ ...(2)

$\dfrac{\partial\phi}{\partial z} = x-y+2z \quad \therefore \quad \phi = \int (x-y+2z)\,dz + f_3(x, y)$

ie., $\phi = xz - yz + z^2 + f_3(x, y)$ ...(3)

Let us choose $f_1(y, z) = y^2 - yz + z^2$,

$f_2(x, z) = \dfrac{x^2}{2} + xz + z^2$, $f_3(x, y) = \dfrac{x^2}{2} + xy + y^2$ from (1), (2) and (3).

Thus the required $\phi = \dfrac{x^2}{2} + xy + xz + y^2 - yz + z^2$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & (x^2 z^2 + z\cos yz) & (2x^2 yz + y\cos yz) \end{vmatrix}$$

$$= i\left\{2x^2 z + (-yz\sin yz + \cos yz) - 2x^2 z - (-yz\sin yz + \cos yz)\right\}$$

$$-j(4xyz - 4xyz) + k(2xz^2 - 2xz^2) = \vec{0}$$

$\therefore$ curl $\vec{F} = \vec{0} \Rightarrow \vec{F}$ **is a potential field.**

Consider $\nabla\phi = \vec{F}$

ie., $\frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k = 2xyz^2 i + (x^2 z^2 + z\cos yz) j + (2x^2 yz + y\cos yz) k$

$\Rightarrow \quad \frac{\partial\phi}{\partial x} = 2xyz^2 \qquad \therefore \quad \phi = \int 2xyz^2 \, dx + f_1(y, z)$

ie., $\phi = x^2 y z^2 + f_1(y, z)$ . . . (1)

$\frac{\partial\phi}{\partial y} = x^2 z^2 + z\cos yz \quad \therefore \quad \phi = \int (x^2 z^2 + z\cos yz)\, dy + f_2(x, z)$

ie., $\phi = x^2 y z^2 + \sin yz + f_2(x, z)$ . . . (2)

$\frac{\partial\phi}{\partial z} = 2x^2 yz + y\cos yz$

$\therefore \quad \phi = \int (2x^2 yz + y\cos yz)\, dz + f_3(x, y)$

ie., $\phi = x^2 yz^2 + \sin yz + f_3(x, y)$ . . . (3)

Let us choose $f_1(y, z) = \sin yz$, $f_2(x, z) = 0$, $f_3(x, y) = 0$ from (1), (2) and (3).

Thus the required $\phi = x^2 y z^2 + \sin yz$

>> We have $\vec{F} = r^{-3}\,\vec{r}$ and we need to show that div $\vec{F} = 0$, and curl $\vec{F} = \vec{0}$.

We have to proceed on the same lines as in Examples 33 and 34 to obtain the result. It is left as an *Exercise* to the reader to establish the results.

>> We need to first establish the result for div $(r^n\,\vec{r})$ and curl $(r^n\,\vec{r})$. These have been worked in Examples 33 and 34.

We have obtained

$$\nabla \cdot (r^n\,\vec{r}) = (n+3)\,r^n \qquad \ldots (1)$$

$$\nabla \times (r^n\,\vec{r}) = \vec{0} \qquad \ldots (2)$$

(a) $\nabla \cdot (r^n\,\vec{r}) = 0$ when $n = -3$ with reference to (1)

$\therefore$ **$r^n\,\vec{r}$ is solenoidal for $n = -3$**

(b) $\nabla \times (r^n\,\vec{r}) = \vec{0}$ with reference to (2)

$\therefore$ **$r^n\,\vec{r}$ is irrotational for all values of n**

(c) Combining these two cases we can easily conclude that,

$r^n\,\vec{r}$ is both solenoidal and irrotational for $n = -3$

These are some properties relating to various meaningful combinations of *gradient, divergence, curl and laplacian*. These are established by taking a general scalar point function or a vector point function.

These have to be **remembered** for working certain types of problems.

VI-3 curl (grad ϕ) = $\vec{0}$, or $\nabla \times (\nabla\phi)$ ...

Proof : Let ϕ be a scalar point function of x, y, z. grad $\phi = \nabla\phi = \dfrac{\partial \phi}{\partial x}\,i + \dfrac{\partial \phi}{\partial y}\,j + \dfrac{\partial \phi}{\partial z}\,k$

$$\text{curl}\,(\text{grad}\,\phi) = \nabla \times (\nabla\phi) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

ie.,
$$= \Sigma \left\{ \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right) \right\} i = \Sigma \left(\frac{\partial^2\phi}{\partial y\,\partial z} - \frac{\partial^2\phi}{\partial z\,\partial y} \right) i = \vec{0}$$

Thus **curl (grad ϕ) = 0, for any scalar function ϕ**

VI-2 div (curl $\vec{A}$) = 0 or $\nabla \cdot (\nabla \times \vec{A}) = 0$

Proof : Let $\vec{A} = a_1\, i + a_2\, j + a_3\, k$ be a vector point function of x, y, z

$$\text{curl}\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \Sigma\, i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right)$$

Now div (curl $\vec{A}$) = $\nabla \cdot (\nabla \times \vec{A})$

$$= \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \Sigma\, i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) = \Sigma \left(\frac{\partial^2 a_3}{\partial x\,\partial y} - \frac{\partial^2 a_2}{\partial x\,\partial z} \right)$$

On expanding we get,

$$\frac{\partial^2 a_3}{\partial x\,\partial y} - \frac{\partial^2 a_2}{\partial x\,\partial z} + \frac{\partial^2 a_1}{\partial y\,\partial z} - \frac{\partial^2 a_3}{\partial y\,\partial x} + \frac{\partial^2 a_2}{\partial z\,\partial x} - \frac{\partial^2 a_1}{\partial z\,\partial y} = 0$$

Thus **div (curl $\vec{A}$) = 0**, for any vector function $\vec{A}$

VI-3 curl (curl $\vec{A}$) = grad (div $\vec{A}$) $- \nabla^2\vec{A}$ or $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$

Proof : Let $\vec{A} = a_1\, i + a_2\, j + a_3\, k$ be a vector point function of x, y, z

$$\therefore \quad \text{curl}\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = \Sigma\, i \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right)$$

Now curl (curl $\vec{A}$) $= \nabla \times (\nabla \times \vec{A})$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\dfrac{\partial a_3}{\partial y} - \dfrac{\partial a_2}{\partial z}\right) & \left(\dfrac{\partial a_1}{\partial z} - \dfrac{\partial a_3}{\partial x}\right) & \left(\dfrac{\partial a_2}{\partial x} - \dfrac{\partial a_1}{\partial y}\right) \end{vmatrix}$$

$$= \Sigma\, i \left\{ \frac{\partial}{\partial y}\left(\frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}\right)\right\}$$

$$= \Sigma\, i \left(\frac{\partial^2 a_2}{\partial y\, \partial x} + \frac{\partial^2 a_3}{\partial z\, \partial x}\right) - \Sigma\, i \left(\frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2}\right), \text{ by rearranging.}$$

(In order to get ∇^2 in the second term and observing that we do not have the second order partial derivative w.r.t x we must think of adding and subtracting the same)

Adding and subtracting $\Sigma\, i \dfrac{\partial^2 a_1}{\partial x^2}$ we get

$$\Sigma\, i \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_2}{\partial y\, \partial x} + \frac{\partial^2 a_3}{\partial z\, \partial x}\right) - \Sigma\, i \left(\frac{\partial^2 a_1}{\partial x^2} + \frac{\partial^2 a_1}{\partial y^2} + \frac{\partial^2 a_1}{\partial z^2}\right)$$

$$= \Sigma\, i \frac{\partial}{\partial x}\left(\frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}\right) - \Sigma \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) a_1\, i$$

$$= \Sigma \frac{\partial}{\partial x} (\text{div } \vec{A})\, i - \nabla^2\, \Sigma\, a_1 i = \text{grad}(\text{div } \vec{A}) - \nabla^2 \vec{A}$$

Thus **curl (curl $\vec{A}$) = grad (div $\vec{A}$) $- \nabla^2 \vec{A}$**

[illegible]

Proof : Let $\vec{A} = a_1 i + a_2 j + a_3 k$ be a vector point function of x, y, z and ϕ be a scalar point function of x, y, z

$\therefore \quad \phi \vec{A} = \phi(a_1 i + a_2 j + a_3 k) = \Sigma\, (\phi a_1)\, i$

Now div $(\phi \vec{A}) = \nabla \cdot (\phi \vec{A})$

$$= \left(\Sigma \frac{\partial}{\partial x} i \right) \cdot \Sigma (\phi a_1) i$$

$$= \Sigma \frac{\partial}{\partial x} (\phi a_1) = \Sigma \left\{ \phi \frac{\partial a_1}{\partial x} + \frac{\partial \phi}{\partial x} a_1 \right\}$$

ie., $$\text{div} (\phi \vec{A}) = \phi \Sigma \frac{\partial a_1}{\partial x} + \Sigma \frac{\partial \phi}{\partial x} i \cdot \Sigma a_1 i$$

(**Note** : *The second term which is of the form* $\Sigma A_1 B_1$ *is being written as* $\Sigma A_1 i \cdot \Sigma B_1 i$)

Thus **div** $(\phi \vec{A}) = \phi (\text{div } \vec{A}) + \text{grad } \phi \cdot \vec{A}$

Proof : Let ϕ and $\vec{A} = a_1 i + a_2 j + a_3 k$ be respectively scalar and vector point functions of x, y, z

$$\therefore \quad \phi \vec{A} = (\phi a_1) i + (\phi a_2) j + (\phi a_3) k$$

Now $$\text{curl} (\phi \vec{A}) = \nabla \times (\phi \vec{A}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi a_1 & \phi a_2 & \phi a_3 \end{vmatrix}$$

ie., $$= \Sigma i \left\{ \frac{\partial}{\partial y} (\phi a_3) - \frac{\partial}{\partial z} (\phi a_2) \right\}$$

$$= \Sigma i \left\{ \left(\phi \frac{\partial a_3}{\partial y} + \frac{\partial \phi}{\partial y} a_3 \right) - \left(\phi \frac{\partial a_2}{\partial z} + \frac{\partial \phi}{\partial z} a_2 \right) \right\}$$

$$= \phi \Sigma \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) i + \Sigma \left(\frac{\partial \phi}{\partial y} a_3 - \frac{\partial \phi}{\partial z} a_2 \right) i$$

$$= \phi \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \phi (\nabla \times \vec{A}) + \nabla \phi \times \vec{A}$$

Thus $\quad \textbf{curl}\ (\phi\ \vec{A}) = \phi\ (\textbf{curl}\ \vec{A}) + \nabla\phi \times \vec{A}$

Remark : *V.I - 4 and 5 when presented interms of* ∇ *are in a format similar to the product rule of differentiation.*

[illegible]

[illegible]

Proof : Let $\vec{A} = a_1\, i + a_2\, j + a_3\, k$ and $\vec{B} = b_1\, i + b_2\, j + b_3\, k$, be two vector point functions of $x,\ y,\ z$

$$\therefore \quad \vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \Sigma\ i\,(a_2\, b_3 - a_3\, b_2)$$

Now $\quad \text{div}\ (\vec{A} \times \vec{B}) = \nabla \cdot (\vec{A} \times \vec{B})$

$$= \left(\Sigma\ \frac{\partial}{\partial x}\ i\right) \cdot\ \Sigma\ i\,(a_2\, b_3 - a_3\, b_2)$$

$$= \Sigma\ \frac{\partial}{\partial x}\ (a_2\, b_3 - a_3\, b_2)$$

$$= \Sigma \left(a_2\ \frac{\partial b_3}{\partial x} + b_3\ \frac{\partial a_2}{\partial x} - a_3\ \frac{\partial b_2}{\partial x} - b_2\ \frac{\partial a_3}{\partial x}\right)$$

On expanding we get

$$\left(a_2\ \frac{\partial b_3}{\partial x} + b_3\ \frac{\partial a_2}{\partial x} - a_3\ \frac{\partial b_2}{\partial x} - b_2\ \frac{\partial a_3}{\partial x}\right) + \left(a_3\ \frac{\partial b_1}{\partial y} + b_1\ \frac{\partial a_3}{\partial y} - a_1\ \frac{\partial b_3}{\partial y} - b_3\ \frac{\partial a_1}{\partial y}\right)$$

$$+ \left(a_1\ \frac{\partial b_2}{\partial z} + b_2\ \frac{\partial a_1}{\partial z} - a_2\ \frac{\partial b_1}{\partial z} - b_1\ \frac{\partial a_2}{\partial z}\right)$$

(**Note:** *We have to focus our attention on the* R.H.S *of the desired result and accordingly plan for rearranging the terms. Since we have dot products with* $\vec{B}$ *and* $\vec{A}$, *naturally we should have the first term lead by* b_1 *and second term lead by* a_1 *with the summation notation*)

$$ie., \quad = \Sigma\ b_1 \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) - \Sigma\ a_1 \left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}\right)$$

$$= (\Sigma\ b_1\, i) \cdot\ \Sigma \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}\right) i\ - (\Sigma\ a_1\, i) \cdot \left(\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}\right) i$$

$(\because \quad \Sigma A_1 B_1 = \Sigma A_1 i \cdot \Sigma B_1 i)$

$$= (\Sigma b_1 i) \cdot \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} - (\Sigma a_1 i) \cdot \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Thus $\quad \textbf{div} \, (\vec{A} \times \vec{B}) = \vec{B} \cdot \textbf{curl} \, \vec{A} - \vec{A} \cdot \textbf{curl} \, \vec{B}$

Note : *We work a few problems by using vector identities. Some problems can be done without using the vector identities but the same becomes very simple by the use of vector identities. In some of the problems we have to necessarily use the identities to arrive at the desired result.*

>> Let $\phi(x, y, z)$ be the scalar point function and $\vec{A}(x, y, z)$ be the vector point function. We have to show that grad ϕ is irrotational and curl $\vec{A}$ is solenoidal. That is to prove that

$$\text{curl}\,(\text{grad}\,\phi) = 0 \quad \text{and} \quad \text{div}\,(\text{curl}\,\vec{A}) = 0$$

We only need to establish the vector identities V.I - 1 and V.I - 2.

>> $\phi(x, y, z)$ is a harmonic function implies that it satisfies the Laplace's equation $\nabla^2 \phi = 0$.

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

We have to show that div (grad ϕ) = 0 and curl (grad ϕ) = $\vec{0}$

$$\text{div}\,(\text{grad}\,\phi) = \nabla \cdot \nabla\phi = \left(\Sigma \frac{\partial}{\partial x} i\right) \cdot \left(\Sigma \frac{\partial \phi}{\partial x} i\right) = \Sigma \frac{\partial^2 \phi}{\partial x^2}$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

By using (1) we have div (grad ϕ) = 0 and hence **grad ϕ is solenoidal.**

Also, establishing curl (grad ϕ) = $\vec{0}$ is nothing but establishing V.I - 1

>> We have to prove that curl ($\phi\nabla\phi$) = $\vec{0}$.

ie., to prove that $\nabla \times (\phi \nabla\phi) = \vec{0}$.

We have the vector identity (V.I - 5)

$$\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + \nabla\phi \times \vec{A} \qquad \textit{(can be assumed)}$$

Taking $\vec{A} = \nabla\phi$ we have

$$\nabla \times (\phi\nabla\phi) = \phi \nabla \times (\nabla\phi) + \nabla\phi \times \nabla\phi$$

$\nabla \times (\phi \nabla\phi) = \vec{0} + \vec{0} = \vec{0}$, because the first term is zero by the vector identity (V.I - 1) and the second term is zero since $\vec{V} \times \vec{V}$ is $\vec{0}$ for any vector $\vec{V}$

Thus $\nabla \times (\phi \nabla\phi) = \vec{0} \Rightarrow$ **$\phi \nabla\phi$ is irrotational.**

Aliter : *(Without using the vector identities)*

$$\nabla \times (\phi \nabla\phi) = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(\phi \dfrac{\partial\phi}{\partial x}\right) & \left(\phi \dfrac{\partial\phi}{\partial y}\right) & \left(\phi \dfrac{\partial\phi}{\partial z}\right) \end{vmatrix}$$

$$= \Sigma\, i \left\{ \frac{\partial}{\partial y}\left(\phi \frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\phi \frac{\partial\phi}{\partial y}\right) \right\}$$

$$= \Sigma\, i \left\{ \left(\phi \frac{\partial^2 \phi}{\partial y\, \partial z} + \frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial z} - \phi \frac{\partial^2 \phi}{\partial z\, \partial y} - \frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial y} \right) \right\} = \vec{0}$$

$\therefore$ **$\phi\ \nabla\phi$ is irrotational.**

>> We have the vector identity (V.I - 4)

$$\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + \nabla\phi \cdot \vec{A}$$

Taking $\phi = u$ and $\vec{A} = \nabla v$ we obtain

$$\nabla \cdot (u\ \nabla v) = u\ \{\nabla \cdot (\nabla v)\} + \nabla u \cdot \nabla v$$

ie., $\nabla \cdot (u\ \nabla v) = u\ \nabla^2 v + \nabla u \cdot \nabla v$ . . . (1)

Similarly we have,

$$\nabla \cdot (v \nabla u) = v \nabla^2 u + \nabla v \cdot \nabla u \qquad \ldots (2)$$

$\therefore$ (1) − (2) will give us

$$\nabla \cdot (u \nabla v) - \nabla \cdot (v \nabla u) = u \nabla^2 v - v \nabla^2 u$$

Thus $\nabla \cdot (u \nabla v - v \nabla u) = u \nabla^2 v - v \nabla^2 u \qquad \ldots (3)$

Further if u and v are harmonic functions then $\nabla^2 u = 0$, $\nabla^2 v = 0$

$\therefore$ (3) becomes $\nabla \cdot (u \nabla v - v \nabla u) = 0$

$\Rightarrow$ $u \nabla v - v \nabla u$ **is solenoidal.**

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55. *If $\vec{F_1}$ and $\vec{F_2}$ are irrotational, prove that $\vec{F_1} \times \vec{F_2}$ is solenoidal.*

>> $\vec{F_1}$ and $\vec{F_2}$ are irrotational by data.

$\Rightarrow$ curl $\vec{F_1} = \vec{0}$ and curl $\vec{F_2} = \vec{0} \qquad \ldots (1)$

We have to prove that div $(\vec{F_1} \times \vec{F_2}) = 0$

We have the vector identity (V.I - 6),

div $(\vec{A} \times \vec{B}) = \vec{B} \cdot$ curl $\vec{A} - \vec{A} \cdot$ curl $\vec{B}$ *(assumed)*

$\therefore$ div $(\vec{F_1} \times \vec{F_2}) = \vec{F_2} \cdot$ curl $\vec{F_1} - \vec{F_1} \cdot$ curl $\vec{F_2}$

ie., div $(\vec{F_1} \times \vec{F_2}) = \vec{F_2} \cdot \vec{0} - \vec{F_1} \cdot \vec{0} = 0$, by using (1)

$\therefore$ div $(\vec{F_1} \times \vec{F_2}) = 0 \Rightarrow \vec{F_1} \times \vec{F_2}$ **is solenoidal.**

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56. *If $u\vec{F} = \nabla v$, prove that $\vec{F}$ and curl $\vec{F}$ are at right angles.*

>> $\vec{F} = \frac{1}{u} \nabla v$ and we have to prove that $\vec{F} \cdot$ curl $\vec{F} = 0$

We shall first find curl $\vec{F}$ where $\vec{F}$ is of the form $\phi \vec{A}$
Let us consider the vector identity (V.I - 5)

$$\nabla \times (\phi \vec{A}) = \phi (\nabla \times \vec{A}) + \nabla \phi \times \vec{A}$$

$$\therefore \quad \nabla \times \left(\frac{1}{u} \nabla v \right) = \frac{1}{u} \left\{ \nabla \times (\nabla v) \right\} + \nabla \left(\frac{1}{u} \right) \times \nabla v$$

The first term in the R.H.S of this equation is zero by a vector identity curl (grad ϕ) = 0 (V.I - 1)

$$\therefore \quad \nabla \times \left(\frac{1}{u} \nabla v\right) = \nabla\left(\frac{1}{u}\right) \times \nabla v \quad ie., \ \text{curl } \vec{F} = \nabla\left(\frac{1}{u}\right) \times \nabla v$$

Now $\vec{F} \cdot \text{curl } \vec{F} = \left(\frac{1}{u} \nabla v\right) \cdot \left\{\nabla\left(\frac{1}{u}\right) \times \nabla v\right\}$

R.H.S of this equation is a scalar triple product or the box product of three vectors.

ie., $\vec{F} \cdot \text{curl } \vec{F} = \left[\frac{1}{u} \nabla v, \nabla\left(\frac{1}{u}\right), \nabla v\right]$

ie., $\vec{F} \cdot \text{curl } \vec{F} = \frac{1}{u}\left[\nabla v, \nabla\left(\frac{1}{u}\right), \nabla v\right] = 0$

(**Note** : *Since the box product of three vectors is equal to coefficient determinant, we have removed* $\frac{1}{u}$ *as a commmon factor from the first row*)

Hence $\vec{F} \cdot \text{curl } \vec{F} = 0$, since two vectors are identical in the box product.

Thus $\vec{F}$ **is perpendicular to curl** $\vec{F}$

57. By using vector identities prove that

(a) $\nabla \cdot (r^n \vec{r}) = (n+3) r^n$ (b) $\nabla \times (r^n \vec{r}) = \vec{0}$

>> (a) We have $\nabla \cdot (\phi \vec{A}) = \phi (\nabla \cdot \vec{A}) + \nabla\phi \cdot \vec{A}$

$$\therefore \quad \nabla \cdot (r^n \vec{r}) = r^n (\nabla \cdot \vec{r}) + \nabla r^n \cdot \vec{r} \quad \ldots (1)$$

Now $\nabla \cdot \vec{r} = \left(\Sigma \frac{\partial}{\partial x} i\right) \cdot (\Sigma x i) = 1 + 1 + 1 = 3$

Also $\nabla r^n = n r^{n-2} \vec{r}$ (*Refer Example - 32*)

$$\therefore \quad \nabla r^n \cdot \vec{r} = n r^{n-2} (\vec{r} \cdot \vec{r}) = n r^{n-2} |\vec{r}|^2 = n r^{n-2} r^2 = n r^n$$

Using these in the R.H.S of (1) we get,

$$\nabla \cdot (r^n \vec{r}) = 3 r^n + n r^n$$

Thus $\nabla \cdot (r^n \vec{r}) = (n+3) r^n$

(b) We have $\nabla \times (\phi \vec{A}) = \phi (\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$

$$\therefore \quad \nabla \times (r^n \vec{r}) = r^n (\nabla \times \vec{r}) + \nabla r^n \times \vec{r} \quad \ldots (2)$$

Now, $\nabla \times \vec{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$

Also $\nabla r^n \times \vec{r} = (n r^{n-2}\,\vec{r}) \times \vec{r} = n r^{n-2}\,(\vec{r} \times \vec{r}) = \vec{0}$

Thus from (2), $\nabla \times (r^n\,\vec{r}) = \vec{0}$

>> $\nabla (r^n) = n\, r^{n-2}\,\vec{r}$ *(Refer Example - 32)*

Also $\nabla^2 (r^n) = \nabla \cdot (\nabla r^n) = \nabla \cdot (n\, r^{n-2}\,\vec{r})$

Consider $\nabla \cdot (\phi\,\vec{A}) = \phi\,(\nabla \cdot \vec{A}) + \nabla\phi \cdot \vec{A}$ (V.I - 4)

$\therefore \quad \nabla \cdot (n r^{n-2}\,\vec{r}) = n \left\{ r^{n-2}\,\nabla \cdot \vec{r} + \nabla (r^{n-2}) \cdot \vec{r} \right\}$

ie., $= n \left\{ 3r^{n-2} + (n-2)\, r^{n-4}\,\vec{r} \cdot \vec{r} \right\} \quad \because \nabla \cdot \vec{r} = 3$

$= n \left\{ 3r^{n-2} + (n-2)\, r^{n-4} r^2 \right\} \quad \because \vec{r} \cdot \vec{r} = r^2$

$= n \left\{ 3r^{n-2} + (n-2)\, r^{n-2} \right\} = n\, r^{n-2} \left\{ 3 + (n-2) \right\}$

Thus $\boldsymbol{\nabla^2 (r^n) = n\,(n+1)\, r^{n-2}}$

>> $\vec{A}$ is solenoidal $\Rightarrow$ div $\vec{A} = 0$ . . . (1)

We have the vector identity (V.I - 3)

curl (curl $\vec{A}$) = grad (div $\vec{A}$) $- \nabla^2 \vec{A}$

$\therefore$ curl (curl $\vec{A}$) $= -\nabla^2 \vec{A}$, by using (1)

Now curl curl {curl (curl $\vec{A}$)} $= -\nabla^2$ curl curl $\vec{A}$

ie., curl curl {curl (curl $\vec{A}$)} $= -\nabla^2 \left\{ -\nabla^2 \vec{A} \right\}$

Thus **curl curl curl curl $\vec{A} = \nabla^2 \nabla^2 \vec{A}$**

>> $\nabla^2 (r^n \vec{r})$ involves Laplacian of a vector point function and hence we recall the identity

$$\text{curl}\ (\text{curl}\ \vec{A}) = \text{grad}\ (\text{div}\ \vec{A}) - \nabla^2 \vec{A} \qquad \ldots (\text{V.I - 3})$$

$$\therefore \quad \nabla^2 \vec{A} = \text{grad}\ (\text{div}\ \vec{A}) - \text{curl}\ (\text{curl}\ \vec{A})$$

Now $\nabla^2 (r^n \vec{r}) = \text{grad} \{ \text{div}\ (r^n \vec{r}) \} - \text{curl} \{ \text{curl}\ (r^n \vec{r}) \}$

But $\text{div}\ (r^n \vec{r}) = (n + 3)\ r^n$ ... *Refer Example - 33*

and $\text{curl}\ (r^n \vec{r}) = \vec{0}$ ... *Refer Example - 34*

$\therefore \quad \nabla^2 (r^n \vec{r}) = \text{grad} \{ (n + 3)\ r^n \} = (n + 3)\ \text{grad}\ (r^n)$

But $\text{grad}\ (r^n) = n\, r^{n-2}\, \vec{r}$ ... *Refer Example - 32*

Thus $\nabla^2 (r^n \vec{r}) = (n + 3)\ n\, r^{n-2} \vec{r} = n (n + 3)\ r^{n-2} \vec{r}$

Find the unit vector normal to the following surfaces at the indicated points (1 to 3)

1. $x^3 + y^3 + 3xyz = 3$ at $(1, 2, -1)$.
2. $x^2 y + y^2 z + z^2 x = 5$ at $(1, -1, 2)$.
3. $xy + yz + zx = 1$ at $(-1, 2, 3)$

Find the directional derivatives of the following : (4 to 9)

4. $\phi = xy + yz + zx$ at $(1, 2, 3)$ along $3i + 4j + 5k$.
5. $\phi = xy^2 + yz^3$ at $(2, -1, 1)$ along $i + 2j + 2k$.
6. $\phi = e^{2x - y + z}$ at $(1, 1, -1)$ in the direction towards the point $(3, 5, -2)$.
7. $\phi = xy^2 + yz^3$ at $(2, -1, 1)$ along the (a) x–axis (b) direction which makes equal angles with the coordinate axes (c) normal to the surface $xy + yz + zx = 3$ at the point $(1, 1, 1)$.
8. $\phi = x^2 y + y^2 z + z^2 x$ at $(1, 1, 1)$ along the tangent to the curve $\vec{r} = t\, i + t^2 j + t^3 k$.
9. $\phi = e^{2x} \cos yz$ at the origin in the direction of the tangent to the curve $x = a \sin t$, $y = a \cos t$, $z = at$ at $t = \pi/4$.

10. In which direction the directional derivative of the function $x^2 y^2 z^3$ is maximum at the point $(1, -1, 2)$? Find the magnitude of this maximum.

11. If the directional derivative of the function $\phi = axy^2 + byz + cz^2 x^3$ at $(1, 2, -1)$ has a maximum magnitude of 64 units in the direction parallel to the z–axis show that the values of a, b, c satisfy the equation $a + b + c = 22$.

12. Find the angle between the surfaces $x \log z = y^2 - 1$ and $x^2 y + z = 2$ at the point $(1, 1, 1)$ and also find the angle between the normals to the surface $x \log z = y^2 - 1$ at the points $(1, 1, 1)$ and $(2, 1, 1)$.

13. Show that the surfaces $4x^2 + z^3 = 4$ and $5x^2 - 2yz - 9x = 0$ intersect each other orthogonally at the point $(1, -1, 2)$.

14. Find the value of the constants a and b such that the surface $x^2 + ayz = 3x$ and $bx^2 y + z^3 = (b - 8)\, y$ intersect each other at right angles at the point $(1, 1, -2)$.

15. Find grad (div $\vec{A}$) and div (curl $\vec{A}$) for the vector $\vec{A} = x^2 i + 3yj + x^3 k$.

16. If $\vec{A} = x y^2 i + 2x^2 yz\, j - 3y^2 z\, k$ find div $\vec{A}$, curl $\vec{A}$ and div (curl $\vec{A}$) at the point $(2, 1, 1)$.

17. If $\vec{A} = \Sigma\, (y^2 + z^2 - x^2)\, i$ prove that div $\vec{A} = -2\, \Sigma\, x$, curl $\vec{A} = 2\, \Sigma\, (y - z)\, i$.

18. If $\phi_1 = x + y + z$, $\phi_2 = x^2 + y^2 + z^2$, $\phi_3 = xy + yz + zx$ prove that the scalar triple product of the vectors $\nabla\phi_1$, $\nabla\phi_2$, $\nabla\phi_3$ is zero.

19. If $\vec{A} = x^2 y\, i - 2xz\, j + 2yz\, k$, find curl (curl $\vec{A}$) and verify that curl (curl $\vec{A}$) = grad (div $\vec{A}$) $- \nabla^2 \vec{A}$.

20. If $\phi = xyz$ and $\vec{A} = x^2 yz\, i + y^2 zx\, j + z^2 xy\, k$ verify that

$$\nabla \cdot (\phi \vec{A}) = \phi\, (\nabla \cdot \vec{A}) + \nabla\phi \cdot \vec{A}$$

21. If $\vec{A} = \Sigma\, x^2 i$, $\vec{B} = \Sigma\, yz\, i$ verify that

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) = 0.$$

Prove the following

22. $\nabla \cdot \left\{ \vec{r}\, \nabla \left(\frac{1}{r^3} \right) \right\} = \frac{3}{r^4}$

23. $\nabla \left\{ \nabla \cdot \left(\frac{\vec{r}}{r} \right) \right\} = -\frac{2\vec{r}}{r^3}$

24. $\nabla^2 (\log r) = \frac{1}{r^2}$

25. $\nabla^2 \left(\frac{x}{r^3} \right) = 0$

26. $\nabla^2 (fg) = f \nabla^2 g + g \nabla^2 f + 2 \nabla f \cdot \nabla g$

27. Find the value of the constant 'a' such that the vector function $\vec{A} = y(ax^2 + z)\, i + x(y^2 - z^2)\, j + 2xy(z - xy)\, k$ is solenoidal. For this value of 'a' show that curl $\vec{A}$ is also solenoidal.

28. Show that the following vector field is irrotational.
$\vec{F} = (\sin y + z)\, i + (x \cos y - z)\, j + (x - y)\, k$
Also find the scalar function ϕ such that $\vec{F} = \nabla\phi$.

29. Find the values of the constants a, b, c such that
$\vec{F} = (x + 2y + az)\, i + (bx - 3y - z)\, j + (4x + cy + 2z)\, k$ is conservative.
Also find its scalar potential.

30. Prove that $\nabla\phi_1 \times \nabla\phi_2$ is irrotational.

1. $\frac{-i + 3j + 2k}{\sqrt{14}}$

2. $\frac{2i - 3j + 5k}{\sqrt{38}}$

3. $\frac{5i + 2j + k}{\sqrt{30}}$

$\frac{46}{5\sqrt{2}}$

5. $-\frac{11}{3}$

6. $\frac{-e^2}{\sqrt{21}}$

7. (a) 1 (b) and (c) : $\frac{-5}{\sqrt{3}}$

8. $\frac{18}{\sqrt{14}}$

9. 1

10. $4(4i - 4j + 3j)$, $4\sqrt{41}$

11. $a = 6,\ b = 24,\ c = -8$

12. $\cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$, $\cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$

14. $a = -1,\ b = 2$

15. $2i$, 0

16. 6, $-11i + 4j$, 0

19. $2(x + 1)\, j$

27. $a = -2$

28. $\phi = x \sin y + xz - yz$

29. $a = 4,\ b = 2,\ c = -1$; $\phi = \frac{x^2}{2} + 2xy + 4xz - \frac{3y^2}{2} - yz + z^2$

Let the coordinates of any point P in space be (x, y, z) in the cartesian system. Suppose x, y, z are expressible interms of new coordinates (u_1, u_2, u_3), we can say that x, y, z are functions of u_1, u_2, u_3. Let us suppose that we are also in a position to express u_1, u_2, u_3 in terms of x, y, z by solving /eliminating. Then the coordinates (u_1, u_2, u_3) are known as ***curvilinear coordinates*** of the point P, where it is assumed that the correspondence between (x, y, z) and (u_1, u_2, u_3) is unique.

The surfaces $u_1 = c_1$ and $u_2 = c_2$, $u_3 = c_3$, c_1, c_2, c_3 being constants, are called ***coordinate surfaces*** and the intersection of each pair of these surfaces give rise to curves called ***coordinate curves.***

A system of curvilinear coordinates is said to be ***orthogonal*** if at each point the tangents to the coordinate curves are mutually perpendicular.

Suppose $\vec{r} = xi + yj + zk$ be the position vector of a point in space, we have $\vec{r} = \vec{r}(u_1, u_2, u_3)$.

$\dfrac{\partial \vec{r}}{\partial u_1}, \dfrac{\partial \vec{r}}{\partial u_2}, \dfrac{\partial \vec{r}}{\partial u_3}$ are called the ***tangent vectors*** to the coordinate curves and the ***unit tangent vectors*** in the same direction are respectively

$$\hat{e}_1 = \frac{\partial \vec{r}}{\partial u_1} \Big/ \left|\frac{\partial \vec{r}}{\partial u_1}\right|, \quad \hat{e}_2 = \frac{\partial \vec{r}}{\partial u_2} \Big/ \left|\frac{\partial \vec{r}}{\partial u_2}\right|, \quad \hat{e}_3 = \frac{\partial \vec{r}}{\partial u_3} \Big/ \left|\frac{\partial \vec{r}}{\partial u_3}\right|$$

The quantities $h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|, h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|, h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$ are called ***scale factors.***

For the ***orthogonality of the curvilinear coordinate system we must have***

$$\hat{e}_1 \cdot \hat{e}_2 = 0, \quad \hat{e}_2 \cdot \hat{e}_3 = 0, \quad \hat{e}_3 \cdot \hat{e}_1 = 0.$$

These are analogous to the property of basic unit vectors in the cartesian system

$$i \cdot j = 0, \quad j \cdot k = 0, \quad k \cdot i = 0$$

We have $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3, \ \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \ \hat{e}_3 \times \hat{e}_1 = \hat{e}_2$.

Thus $\hat{e}_1, \hat{e}_2, \hat{e}_3$ form a right handed system of vectors. If $\vec{A}$ is any vector in the orthogonal curvilinear coordinate system then

$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$ where A_1, A_2, A_3 are scalar functions of u_1, u_2, u_3.

In addition to the well acquainted rectangular cartesian coordinates (x, y, z) we introduce two new set of coordinates.

(i) ***Cylindrical polar coordinates*** (ρ, ϕ, z) given by the transformation:

$$x = \rho \cos \phi, \ y = \rho \sin \phi, \ z = z$$

(ii) ***Spherical polar coordinates*** (r, θ, ϕ) given by the transformation :

$$x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi, \ z = r \cos \theta.$$

We are familiar with the vector differential operator $\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$

and the Laplacian operator $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ operated on scalar and vector point functions.

If ψ is a scalar function and $\vec{A}$ is a vector function of x, y, z we know that :

$$\nabla \psi = \text{grad}\, \psi, \quad \nabla \cdot \vec{A} = \text{div}\, \vec{A}, \nabla \times \vec{A} = \text{curl}\, \vec{A}, \quad \nabla^2 \psi = \text{Laplacian of } \psi.$$

In this topic we obtain expressions for these in a general curvilinear coordinate system with special reference to the cylindrical system and spherical system as particular cases.

The cylindrical polar coordinates (ρ, ϕ, z) is regarded as a particular case of the general orthogonal curvilinear coordinates (u_1, u_2, u_3) by setting $(u_1, u_2, u_3) = (\rho, \phi, z)$ and are related to the cartesian coordinates (x, y, z) by the transformation :

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z.$$

Thus $\vec{r} = x\,i + y\,j + z\,k$ becomes

$$\vec{r} = \rho\cos\phi\, i + \rho\sin\phi\, j + zk$$

We have by the definition of scale factors,

$$h_1 = \left|\frac{\partial \vec{r}}{\partial \rho}\right| = \left|\cos\phi\, i + \sin\phi\, j + 0k\right| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \left|-\rho\sin\phi\, i + \rho\cos\phi\, j + 0k\right| = \sqrt{\rho^2(\sin^2\phi + \cos^2\phi)} = \rho$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial z}\right| = \left|0i + 0j + 1k\right| = \sqrt{0^2 + 0^2 + 1^2} = 1$$

Thus $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$ fr the cylindrical system.

We have $(u_1, u_2, u_3) = (r, \theta, \phi)$ and by the transformation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ we have

$$\vec{r} = r\sin\theta\cos\phi\, i + r\sin\theta\sin\phi\, j + r\cos\theta\, k$$

$$h_1 = \left|\frac{\partial \vec{r}}{\partial r}\right| = \left|\sin\theta\cos\phi\, i + \sin\theta\sin\phi\, j + \cos\theta\, k\right|$$

ie., $$= \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial \theta}\right| = \left|r\cos\theta\cos\phi\, i + r\cos\theta\sin\phi\, j - r\sin\theta\, k\right|$$

ie., $$= \sqrt{r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta}$$

$$= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = r$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial \phi}\right| = \left|-r\sin\theta\sin\phi\, i + r\sin\theta\cos\phi\, j + 0k\right|$$

ie., $= \sqrt{r^2 \sin^2\theta\,(\sin^2\phi + \cos^2\phi)} = r\sin\theta$

Thus $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$ for the spherical system.

Note : $\vec{r} = x\,i + y\,j + z\,k$ *will give us* $h_1 = 1 = h_2 = h_3$ *for the cartesian system.*

We have for the cylindrical system $\vec{r} = \rho\cos\phi\, i + \rho\sin\phi\, j + z\,k$

Let $\hat{e}_\rho,\ \hat{e}_\phi,\ \hat{e}_z$ be the basic unit vectors of this system.

They are given by

$$\hat{e}_\rho = \frac{\partial \vec{r}}{\partial \rho} \Big/ \left|\frac{\partial \vec{r}}{\partial \rho}\right| = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial \rho} = (\cos\phi\, i + \sin\phi\, j + 0k), \quad \text{since } h_1 = 1.$$

$$\hat{e}_\phi = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{\rho}(-\rho\sin\phi\, i + \rho\cos\phi\, j + 0k), \quad \text{since } h_2 = 1.$$

ie., $\hat{e}_\phi = -\sin\phi\, i + \cos\phi\, j + 0k$

$$\hat{e}_z = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial z} = \frac{1}{1}(0i + 0j + 1k) = 0i + 0j + 1k, \quad \text{since } h_3 = 1.$$

Now $\hat{e}_\rho \cdot \hat{e}_\phi = -\cos\phi\sin\phi + \sin\phi\cos\phi = 0\ ;\ \hat{e}_\phi \cdot \hat{e}_z = 0,\ \hat{e}_z \cdot \hat{e}_\rho = 0$

Thus the cylindrical system is orthogonal.

We have for the spherical system

$$\vec{r} = r\sin\theta\cos\phi\, i + r\sin\theta\sin\phi\, j + r\cos\theta\, k$$

and let $\hat{e}_r,\ \hat{e}_\theta,\ \hat{e}_\phi$ be the basic unit vectors of this system.

Further we have $h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

Now, $\hat{e}_r = \dfrac{1}{h_1}\dfrac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\, i + \sin\theta\sin\phi\, j + \cos\theta\, k$

$$\hat{e}_\theta = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial \theta} = \frac{1}{r}(r\cos\theta\cos\phi\, i + r\cos\theta\sin\phi\, j - r\sin\theta\, k)$$

ie., $\hat{e}_\theta = \cos\theta\cos\phi\, i + \cos\theta\sin\phi\, j - \sin\theta\, k$

$$\hat{e}_\phi = \frac{1}{h_3}\frac{\partial \vec{r}}{\partial \phi} = \frac{1}{r\sin\theta}(-r\sin\theta\sin\phi\, i + r\sin\theta\cos\phi\, j + 0k)$$

ie., $\hat{e}_\phi = -\sin\phi\, i + \cos\phi\, j + 0k$

Now $\hat{e}_r \cdot \hat{e}_\theta = \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) - \sin\theta\cos\theta = 0$

$$\hat{e}_\theta \cdot \hat{e}_\phi = -\cos\theta\cos\phi\sin\phi + \cos\theta\sin\phi\cos\phi = 0$$

$$\hat{e}_\phi \cdot e_r = -\sin\theta\cos\phi\sin\phi + \sin\theta\cos\phi\sin\phi = 0$$

Thus the spherical system is orthogonal.

We have $\vec{r} = \vec{r}(u_1, u_2, u_3)$

$\therefore$ $d\vec{r} = \frac{\partial \vec{r}}{\partial u_1}du_1 + \frac{\partial \vec{r}}{\partial u_2}du_2 + \frac{\partial \vec{r}}{\partial u_3}du_3$ (*total derivative*)

ie., $d\vec{r} = h_1\, du_1\, \hat{e}_1 + h_2\, du_2\, \hat{e}_2 + h_3\, du_3\, \hat{e}_3$

For a curve in space through the point P the *arc length* ds is given by tl . relation $ds = |\vec{dr}|$

ie., $ds = \sqrt{h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2}$

or $ds^2 = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2$

The vector $\vec{dr}$ along the u_1 curve for which u_2 and u_3 are constants is given by $\vec{dr} = h_1\, du_1\, \hat{e}_1$ since $du_2 = 0 = du_3$.

Similarly along the u_2 curve and u_3 curve we have $h_2\, du_2\, \hat{e}_2$ and $h_3\, du_3\, \hat{e}_3$ respectively.

The volume of the rectangular parallelopiped formed by these is called the volume element dV at P in the orthogonal curvilinear coordinate system. Using the geometrical meaning of the scalar triple product of vectors we have,

$$dV = h_1\, du_1\, \hat{e}_1 \cdot (h_2\, du_2\, \hat{e}_2 \times h_3\, du_3\, \hat{e}_3) = h_1\, h_2\, h_3\, du_1\, du_2\, du_3 \left\{\hat{e}_1 \cdot (\hat{e}_2 \times \hat{e}_3)\right\}.$$

But $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$ and $\hat{e}_1 \cdot \hat{e}_1 = 1$

$\therefore$ $dV = h_1\, h_2\, h_3\, du_1\, du_2\, du_3$

Thus $ds^2 = h_1^2\, du_1^2 + h_2^2\, du_2^2 + h_3^2\, du_3^2$ and $dV = h_1\, h_2\, h_3\, du_1\, du_2\, du_3$

(i) In the cylindrical system :

$$(u_1, u_2, u_3) = (\rho, \phi, z) \text{ and } h_1 = 1, h_2 = \rho, h_3 = 1$$

$\therefore$ $$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \; ; \; dV = \rho\, d\rho\, d\phi\, dz$$

(ii) In the spherical system : $(u_1, u_2, u_3) = (r, \theta, \phi)$ and

$$h_1 = 1, h_2 = r, h_3 = r\sin\theta.$$

$\therefore$ $$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2, \; dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Consider a scalar point function $\psi(u_1, u_2, u_3)$

Let $$\nabla\psi = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 \qquad \ldots (1)$$

where a_1, a_2, a_3 are to be determined.

We also have $\vec{r} = \vec{r}(u_1, u_2, u_3)$ and as a total derivative,

$$\vec{dr} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

i.e., $$\vec{dr} = h_1\, du_1\, \hat{e}_1 + h_2\, du_2\, \hat{e}_2 + h_3\, du_3\, \hat{e}_3 \qquad \ldots (2)$$

We have the fact that x, y, z are functions of u_1, u_2, u_3 and vice-versa.

We are also familiar with the result : $d\psi = \vec{dr} \cdot \nabla\psi$

Hence $$d\psi = a_1 h_1\, du_1 + a_2 h_2\, du_2 + a_3 h_3\, du_3 \qquad \ldots (3)$$

But as a total derivative we also have from $\psi = \psi(u_1, u_2, u_3)$

$$d\psi = \frac{\partial \psi}{\partial u_1} du_1 + \frac{\partial \psi}{\partial u_2} du_2 + \frac{\partial \psi}{\partial u_3} du_3 \qquad \ldots (4)$$

Equating the R.H.S of (3) and (4) we have,

$$a_1 h_1 = \frac{\partial \psi}{\partial u_1}, \; a_2 h_2 = \frac{\partial \psi}{\partial u_2}, \; a_3 h_3 = \frac{\partial \psi}{\partial u_3}$$

$\therefore$ $$a_1 = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}, \; a_2 = \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}, \; a_3 = \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}$$

Substituting these values in (1) we obtain,

$$\nabla \psi = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}\hat{e}_3 = \Sigma \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1$$

(i) In the cylindrical system :

$$(u_1, u_2, u_3) = (\rho, \phi, z);\ h_1 = 1\ h_2 = \rho,\ h_3 = 1$$

$$\nabla \psi = \frac{\partial \psi}{\partial \rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial \psi}{\partial \phi}\hat{e}_\phi + \frac{\partial \psi}{\partial z}\hat{e}_z$$

(ii) In the spherical system :

$$\nabla \psi = \frac{\partial \psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{e}_\phi$$

Note : *The following vector identities will be useful.*

V.I - 1. $\nabla \times (\nabla \phi) = \vec{0}$

V.I - 2. $\nabla \cdot (\nabla \times \vec{F}) = 0$

V.I - 3. $\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + (\nabla \phi \cdot \vec{A})$

V.I - 4. $\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + (\nabla \phi \times \vec{A})$

V.I - 5. $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

Let $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$

$$\text{div}\,\vec{A} = \nabla \cdot \vec{A} = \nabla \cdot (A_1\hat{e}_1) + \nabla \cdot (A_2\hat{e}_2) + \nabla \cdot (A_3\hat{e}_3) \qquad \ldots (1)$$

We have $\nabla \psi = \Sigma \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1$ ... (2)

$$\therefore \qquad \nabla u_1 = \frac{1}{h_1}\hat{e}_1 + 0 + 0$$

That is $\nabla u_1 = \frac{\hat{e}_1}{h_1},\ \nabla u_2 = \frac{\hat{e}_2}{h_2},\ \nabla u_3 = \frac{\hat{e}_3}{h_3}$

Also $\hat{e}_1 = \hat{e}_2 \times \hat{e}_3 = (h_2 \nabla u_2) \times (h_3 \nabla u_3)$

or $\hat{e}_1 = h_2 h_3 (\nabla u_2 \times \nabla u_3)$ ... (3)

Let us consider only the first term in the R.H.S of (1) and proceed as follows.

$$\nabla \cdot (A_1 \hat{e}_1) = \nabla \cdot \{ A_1 h_2 h_3 (\nabla u_2 \times \nabla u_3) \}, \text{ by using (3).}$$

ie., $$= \nabla \cdot (\phi \vec{a}) \text{ where } \phi = A_1 h_2 h_3, \ \vec{a} = (\nabla u_2 \times \nabla u_3)$$

$$= \phi (\nabla \cdot \vec{a}) + \vec{a} \cdot \nabla \phi \qquad [\text{By using V.I - 3}]$$

$$= A_1 h_2 h_3 \{ \nabla \cdot (\nabla u_2 \times \nabla u_3) \} + (\nabla u_2 \times \nabla u_3) \cdot \nabla (A_1 h_2 h_3)$$

$$= A_1 h_2 h_3 \{ \nabla u_3 \cdot \nabla \times (\nabla u_2) - \nabla u_2 \cdot \nabla \times (\nabla u_3) \} + \frac{\hat{e}_1}{h_2 h_3} \cdot \nabla (A_1 h_2 h_3)$$

[*We have used V.I - 5 for the first term and expression* (3) *for the second term*]

$$= 0 + \frac{\hat{e}_1}{h_2 h_3} \cdot \Sigma \frac{1}{h_1} \frac{\partial}{\partial u_1} (A_1 h_2 h_3) \hat{e}_1$$

[*We have used V.I - 1 for the first term and the expression format of* $\nabla \psi$ *for the second term*]

By expanding R.H.S and taking the dot product we get,

$$\nabla \cdot (A_1 \hat{e}_1) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (A_1 h_2 h_3)$$

$$\because \quad \hat{e}_1 \cdot \hat{e}_1 = 1, \ \hat{e}_1 \cdot \hat{e}_2 = 0, \ \hat{e}_1 \cdot \hat{e}_3 = 0$$

Similarly $$\nabla \cdot (A_2 \hat{e}_2) = \frac{1}{h_2 h_3 h_1} \frac{\partial}{\partial u_2} (A_2 h_3 h_1)$$

$$\nabla \cdot (A_3 \hat{e}_3) = \frac{1}{h_3 h_1 h_2} \frac{\partial}{\partial u_3} (A_3 h_1 h_2)$$

Adding these results we have,

$$\nabla \cdot (A_1 \hat{e}_1) + \nabla \cdot (A_2 \hat{e}_2) + \nabla \cdot (A_3 \hat{e}_3) = \frac{1}{h_1 h_2 h_3} \Sigma \frac{\partial}{\partial u_1} (A_1 h_2 h_3)$$

Using (1) for the L.H.S. we have,

$$\nabla \cdot \vec{A} = \textbf{div}\, \vec{A} = \frac{1}{h_1 h_2 h_3} \Sigma \frac{\partial}{\partial u_1} (A_1 h_2 h_3)$$

Corollary

We deduce expressions for $\nabla \cdot \vec{A}$ in the cylindrical and spherical systems by using the expression for the same in the expanded form :

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right\}$$

We have $(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$, for the cylindrical system. Hence we obtain,

$$\nabla \cdot \vec{A} = \frac{1}{\rho} \left\{ \frac{\partial}{\partial \rho} (\rho A_1) + \frac{\partial}{\partial \phi} (A_2) + \frac{\partial}{\partial z} (\rho A_3) \right\} \qquad \textit{(Cylindrical system)}$$

Next, $(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1, h_2 = r, h_3 = r \sin \theta$ for the spherical system. Hence we obtain,

$$\nabla \cdot \vec{A} = \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} (r^2 \sin \theta A_1) + \frac{\partial}{\partial \theta} (r \sin \theta A_2) + \frac{\partial}{\partial \phi} (r A_3) \right\} \qquad \textit{(Spherical system)}$$

Let $\quad \vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$

$$\text{curl} \vec{A} = \nabla \times \vec{A} = \nabla \times (A_1 \hat{e}_1) + \nabla \times (A_2 \hat{e}_2) + \nabla \times (A_3 \hat{e}_3) \qquad \ldots (1)$$

We have, $\nabla \psi = \Sigma \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 \qquad \therefore \quad \nabla u_1 = \frac{1}{h_1} \hat{e}_1$

ie., $\quad \hat{e}_1 = h_1 \nabla u_1 \qquad \ldots (2)$

.We shall consider only the first term in the R.H.S of (1) and proceed as follows.

$\nabla \times (A_1 \hat{e}_1) = \nabla \times (A_1 h_1 \nabla u_1)$, by using (2).

ie., $\quad = \nabla \times (\phi \vec{a})$ *where* $\phi = A_1 h_1$, $\vec{a} = \nabla u_1$

$= \phi (\nabla \times \vec{a}) + \nabla \phi \times \vec{a}$ $\qquad$ *[By using V.I - 4]*

$= A_1 h_1 \{ \nabla \times (\nabla u_1) \} + \nabla (A_1 h_1) \times \nabla u_1$

$= 0 + \nabla (A_1 h_1) \times \nabla u_1 \qquad \because \nabla \times \nabla \phi = 0$

$$= \left\{ \frac{1}{h_1} \frac{\partial}{\partial u_1} (A_1 h_1) \hat{e}_1 + \frac{1}{h_2} \frac{\partial}{\partial u_2} (A_1 h_1) \hat{e}_2 + \frac{1}{h_3} \frac{\partial}{\partial u_3} (A_1 h_1) \hat{e}_3 \right\} \times \left(\frac{\hat{e}_1}{h_1} \right)$$

where we have used the expression format of $\nabla \psi$ in the expanded form and (2). Also, using the fact that

$\hat{e}_1 \times \hat{e}_1 = 0,\ \hat{e}_2 \times \hat{e}_1 = -\hat{e}_3,\ \hat{e}_3 \times \hat{e}_1 = \hat{e}_2$ we have

$$\nabla \times (A_1 \hat{e}_1) = \frac{-\hat{e}_3}{h_1 h_2} \frac{\partial}{\partial u_2}(A_1 h_1) + \frac{\hat{e}_2}{h_1 h_3} \frac{\partial}{\partial u_3}(A_1 h_1)$$

Similarly by symmetry,

$$\nabla \times (A_2 \hat{e}_2) = \frac{-\hat{e}_1}{h_2 h_3} \frac{\partial}{\partial u_3}(A_2 h_2) + \frac{\hat{e}_3}{h_2 h_1} \frac{\partial}{\partial u_1}(A_2 h_2)$$

$$\nabla \times (A_3 \hat{e}_3) = \frac{-\hat{e}_2}{h_3 h_1} \frac{\partial}{\partial u_1}(A_3 h_3) + \frac{\hat{e}_1}{h_3 h_2} \frac{\partial}{\partial u_2}(A_3 h_3)$$

Adding these results, L.H.S becomes $\nabla \times \vec{A}$ according to (1) and R.H.S can be put in the determinant form as follows.

Thus $$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

Corollary

(i) In the cylindrical system : $(u_1, u_2, u_3) = (\rho, \phi, z)$;
$h_1 = 1,\ h_2 = \rho,\ h_3 = 1$ and the basic unit vectors are denoted by $\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z$

$$\therefore \quad \nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{e}_\rho & \rho \hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_1 & \rho A_2 & A_3 \end{vmatrix}$$

(ii) In the spherical system : $(u_1, u_2, u_3) = (r, \theta, \phi)$;
$h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$ and the basic unit vectors are denoted by $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$

$$\therefore \qquad \nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\,\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_1 & r A_2 & r\sin\theta\,A_3 \end{vmatrix}$$

We know that $\nabla^2 \psi = \nabla \quad \nabla \psi$ and we have

$$\nabla \psi = \Sigma \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e}_1 \qquad \ldots (1)$$

Also if $\vec{A} = A_1\,\hat{e}_1 + A_2\,\hat{e}_2 + A_3\,\hat{e}_3$ we have,

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\Sigma \frac{\partial}{\partial u_1}(A_1 h_2 h_3) \qquad \ldots (2)$$

We need to substitute (1) in (2). That is by taking $\vec{A} = \nabla \psi$ which is equivalent to taking $A_1 = \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}$ since $\vec{A} = \Sigma A_1 \hat{e}_1$

$$\therefore \qquad \nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\Sigma \frac{\partial}{\partial u_1}\left(\frac{1}{h_1}\frac{\partial \psi}{\partial u_1} h_2 h_3\right)$$

$$ie., \qquad \nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\Sigma \frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial u_1}\right) \qquad \ldots (3)$$

$$\text{Thus} \quad \nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial u_3}\right)\right]$$

In the cylindrical system we have $(u_1, u_2, u_3) = (\rho, \phi, z)$ and $h_1 = 1, h_2 = \rho, h_3 = 1$.

$$\therefore \quad \nabla^2 \psi = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho \frac{\partial \psi}{\partial z}\right)\right]$$

$$= \frac{1}{\rho}\left[\rho \frac{\partial^2 \psi}{\partial \rho^2} + \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho}\frac{\partial^2 \psi}{\partial \phi^2} + \rho \frac{\partial^2 \psi}{\partial z^2}\right]$$

Thus $$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$\nabla^2 \psi$ in the spherical system

In the spherical system we have $(u_1, u_2, u_3) = (r, \theta, \phi)$ and $h_1 = 1, h_2 = r, h_3 = r \sin\theta$. Substituting in the general expression for $\nabla^2 \psi$ we get,

$$\nabla^2 \psi = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \phi}\right)\right]$$

$$= \frac{1}{r^2 \sin\theta}\left[\sin\theta\left(r^2 \frac{\partial^2 \psi}{\partial r^2} + 2r\frac{\partial \psi}{\partial r}\right) + \sin\theta \frac{\partial^2 \psi}{\partial \theta^2} + \cos\theta \frac{\partial \psi}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial \phi^2}\right]$$

Thus $$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

Results at a glance

1. *Cylindrical system* : $(u_1, u_2, u_3) = (\rho, \phi, z)$; $h_1 = 1, h_2 = \rho, h_3 = 1$
2. *Spherical system* : $(u_1, u_2, u_3) = (r, \theta, \phi)$; $h_1 = 1, h_2 = r, h_3 = r \sin\theta$
3. grad $\psi = \nabla \psi = \frac{1}{h_1} \Sigma \frac{\partial \psi}{\partial u_1} \hat{e}_1$

4. $\operatorname{div} \vec{A} = \nabla \cdot \vec{A} = \dfrac{1}{h_1 h_2 h_3} \Sigma \dfrac{\partial}{\partial u_1} (A_1 h_2 h_3)$

where $\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$

5. $\operatorname{curl} \vec{A} = \nabla \times \vec{A} = \dfrac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$

6. $\nabla^2 \psi = \dfrac{1}{h_1 h_2 h_3} \Sigma \dfrac{\partial}{\partial u_1} \left(\dfrac{h_2 h_3}{h_1} \dfrac{\partial \psi}{\partial u_1} \right)$

Remark : $(u_1 , u_2 , u_3) = (x , y , z)$ *and* $h_1 = h_2 = h_3 = 1$ *for the cartesian system. Results (3) to (6) reduces to the already known definitions in the cartesian system.*